Section 16.9: The Divergence Theorem

Tuesday, May 5, 2015 3:49 PM

Goal: To use the Divergence Theorem to evaluate a Flux integral

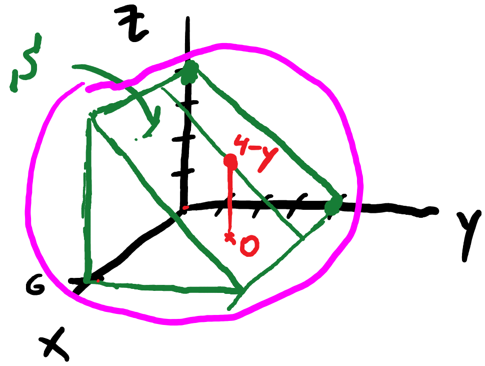
Divergence Theorem : $$\boxed{\iint_S \vec{F}\cdot\vec{N}\,dS = \iiint_E \underbrace{\operatorname{div}\vec{F}\,dV}_{\text{Flux element}}}$$

where E is the solid enclosed by the closed surface S, which is oriented by the outward pointing $\vec{N}$. Also $\vec{F}$ has continuous 1st partials and E smooth enough for the triple integral to exist.

ex $F(x,y,z) = xe^z\,\vec{i} + ye^z\,\vec{j} + e^z\,\vec{k}$.

S: $\boxed{z = 4-y,\ z=0,\ x=0,\ x=6,\ y=0}$ Find flux.

$$\iint_S \vec{F}\cdot\vec{N}\,dS$$



To evaluate directly you need to calculate the flux over each of the five faces of S and sum.

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \boxed{\iiint_E \operatorname{div}\vec{F}\,dV}$$

$$\operatorname{div}\vec{F} = e^z + e^z + e^z = \boxed{3e^z}$$

$$\text{Flux} = \int_0^6 \int_0^4 \int_0^{4-y} 3e^z\,dz\,dy\,dx$$

$$= 3\int_0^6 \int_0^4 \left[e^z\right]_0^{4-y} dy\,dx$$

$$= 3\int_0^6 \int_0^4 \left(e^{4-y} - 1\right) dy\,dx$$

$$= 3\underbrace{\int_0^6 dx}_{6} \int_0^4 \left(e^{4-y} - 1\right) dy$$

$$= 18\left[-e^{4-y} - y\right]_0^4$$

$$= 18\left[(-1-4) - (-e^4 - 0)\right]$$

$$= 18\left(e^4 - 5\right)$$

Physical Interpretation of Divergence

Let S be a "small" closed surface containing P. The by Div. Th.

$$\text{Flux through } S = \iint_S \vec{F} \cdot \vec{N}\, dS = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$\text{''} \quad \approx \iiint_E \operatorname{div} \vec{F}(P)\, dV$$

$$\text{''} \quad = \operatorname{div} \vec{F}(P) \iiint_E 1\, dV$$

$$\text{Flux through } S = \operatorname{div} \vec{F}(P) \cdot (\text{Volume of } E)$$

$$\operatorname{div} \vec{F}(P) = \boxed{\frac{\text{Flux through } S}{\text{Volume of } E}}$$

$\operatorname{div} \vec{F}(P)$ = Flux per unit ~~area~~ volume

Note: ① Total divergence in a closed E equals to flow across S.

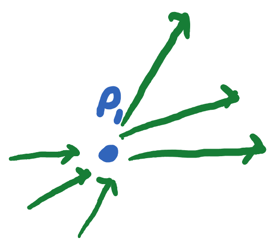
② For closed S, Total flux $= \begin{pmatrix} \text{amount of} \\ \text{flow out of} \\ S \end{pmatrix} - \begin{pmatrix} \text{Amount} \\ \text{of flow} \\ \text{in} \end{pmatrix}$

③ If $\operatorname{div} \vec{F}(P) > 0$, then P is a <u>source</u>

⑤ If $\operatorname{div} \vec{F}(P) < 0$, then P is a <u>sink</u>

⑥ If $\operatorname{div} \vec{F}(P) = 0$, then $\vec{F}$ is incompressible.

(ex) Determine whether $\operatorname{div} \vec{F}(P)$ > 0, < 0, $= 0$

a) 

$\operatorname{div} \vec{F}(P_1) > 0$ (P_1 is a source)

b) 

$\operatorname{div} \vec{F}(P_2) < 0$ (P_2 is a sink)