Section 16.5: Curl and Divergence

Monday, April 27, 2015 8:47 PM

Goal: To calculate the curl and divergence of **F** = <P, Q, R>

Def

① "del," ∇, is differential operator:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$f(x, y, z)$

Gradient: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

$= \langle f_x, f_y, f_z \rangle$

② Divergence of f: $\underbrace{\nabla \cdot \vec{F}}_{\text{div } \vec{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

③ curl of $\vec{F}$: curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

(ex) Find the div $\vec{F}$ and curl $\vec{F}$ where

$\vec{F} = \langle xe^y, xz, ze^y \rangle$

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle x e^{-y}, xz, z e^{y} \right\rangle$$

$$= e^{-y} + 0 + e^{y}$$

$$= e^{-y} + e^{y}$$

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x e^{-y} & xz & z e^{y} \end{vmatrix}$$

$$= (z e^{y} - x)\vec{i} - (0 - 0)\vec{j} + (z + (+ x e^{-y}))\vec{k}$$

$$= (z e^{y} - x)\vec{i} + (z + x e^{-y})\vec{k}$$

Theorem: $\vec{F}$ is conservative $\boxed{\text{iff}}$ $\operatorname{curl}\vec{F} = \vec{0}$

$\uparrow \vec{F} = \nabla f$

$\Rightarrow$ ($\vec{F}$ conserv.)

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \vec{0}.$$

$\Leftarrow$ use Stokes' Theorem

$P \quad Q \quad R$

f $\quad P \; f_x \quad Q \; f_y \quad R \; f_z$

(ex) Determine if $\vec{F} = e^z \vec{i} + 1\vec{j} + xe^z \vec{k}$ is conservative. If so, find a potential function for f.

To show $\vec{F}$ conservative, take $\text{curl}\,\vec{F}$ to show it is $\vec{0}$ [i.e. show $\text{curl}\,\vec{F} = \vec{0}$]

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^z & 1 & xe^z \end{vmatrix} = \vec{0}$$

$$f(x,y,z) = \int e^z \, dx$$

$$= \boxed{xe^z + g(y,z)}$$

$$\int g_y(y,z)\,dy = \int 1 \, dy$$

$$g(y,z) = \boxed{y + h(z)}$$

$$f(x,y,z) = xe^z + y + h(z)$$

$$f_z = \boxed{\cancel{xe^z} + h'(z) = \cancel{xe^z}}$$

$$h'(z) = 0$$

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$$h(z) = k, \quad k \text{ is a constant}$$

$$\boxed{f(x,y,z) = xe^{z} + y + k}$$