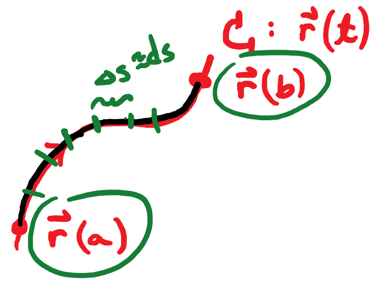
Section 16.2: Line Integrals

Thursday, April 16, 2015 4:34 PM

Goal: To set up and evaluate line integrals

Recall the arc length formula

$$\text{Length} = \int_a^b \underbrace{\overbrace{\|\vec{r}\,'(t)\|}^{\frac{ds}{dt}}\, dt}_{ds} = \int_C 1\, ds$$



idea: Stick in a function!

Line Integral w.r.t arclength

$$\boxed{\int_C f(x,y,z)\, ds} = \int_a^b f(x(t), y(t), z(t))\, \|\vec{r}\,'(t)\|\, dt$$

Assumptions: C is smooth or piecewise smooth,

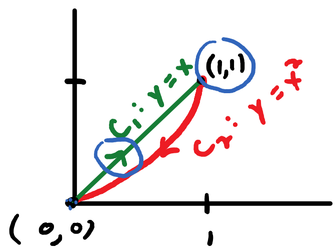
$(\vec{r}\,'(t) \neq \vec{0}$ and $\vec{r}\,'$ is continuous$)$

or piece-wise smooth.

Note: say $f(x,y,z)$ (λ) is a density function of a wire (in $\frac{\text{mass}}{\text{length}}$). $\int_C \underbrace{f(x,y,z)\, ds}_{\text{mass element}}$

total mass of wire.

ex Evaluate $\boxed{\int_C 12x\,ds}$ where $C = C_1 \cup C_2$ is:



Note: $\int_{C = C_1 \cup C_2} f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$

$C_1: \; y = x$ ← t

$x = t, \quad y = t$

$\vec{r}(t) = t\vec{i} + t\vec{j}, \quad 0 \le t \le 1$

$ds = \|\vec{r}\,'(t)\|\,dt$

$\vec{r}\,'(t) = 1\vec{i} + 1\vec{j}$

$\|\vec{r}\,'(t)\| = \sqrt{1+1} = \sqrt{2}$

$\int_{C_1} 12x\,ds = \int_0^1 12t\,\|\vec{r}\,'(t)\|\,dt$

$= \int_0^1 12t\,\sqrt{2}\,dt$

$= 12\sqrt{2}\int_0^1 t\,dt$

$= \frac{12\sqrt{2}}{2}\left[t^2\right]_0^1$

$= 6\sqrt{2}$

$C_2: \; y = x^2$

$x = t \quad y = t^2$

$\vec{r}(t) = t\vec{i} + t^2\vec{j}, \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \vec{i} + 2t\vec{j}$

$\|\vec{r}\,'(t)\| = \sqrt{1+4t^2}$

$\int_{C_2} 12x\,ds = \int_0^1 12t\sqrt{1+4t^2}\,dt$

$= \frac{12}{8}\int_0^1 (8t)\sqrt{1+4t^2}\,dt$

$= \frac{3}{2}\cdot\frac{2}{3}\left[(1+4t^2)^{\frac{3}{2}}\right]_0^1$

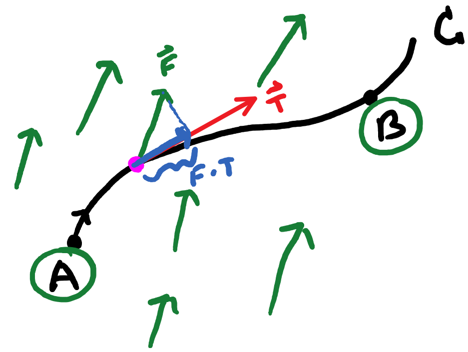
$= 5^{\frac{3}{2}} - 1$

$\int_C 12x\,ds = 6\sqrt{2} + 5\sqrt{5} - 1$

Note: orientation of C doesn't matter when integrating w.r.t. arc length.

Line Integral w.r.t. a Vector Field

Question: Suppose a particle moves through a vector field **F** along C, then what does the following integral represent?



$$W = \int_C \overbrace{\underbrace{\vec{F}\cdot\vec{T}}_{\text{force on particle in direction of motion}}\, ds}^{(\text{Force})\cdot\text{distance} = W}$$

= Total work done by $\vec{F}$ in moving the particle from A to B

Line Integral of a vector field

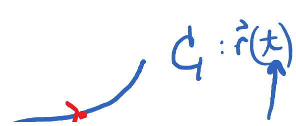
$$\boxed{\int_C \vec{F}\cdot\vec{T}\, ds} = \int_a^b \vec{F}\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\,\|\vec{r}\,'(t)\|\, dt$$

$$= \boxed{\int_a^b \vec{F}\cdot\underbrace{\vec{r}\,'(t)\, dt}_{d\vec{r}}} \quad \text{calculation formula}$$

$$\left\{ \begin{array}{l} \dfrac{d\vec{r}}{dt} = \vec{r}\,'(t) \\ d\vec{r} = \vec{r}\,'(t)\, dt \end{array} \right.$$

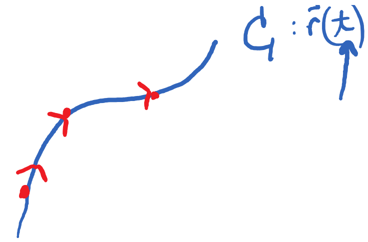
$$= \boxed{\int_C \vec{F}\cdot d\vec{r}}$$

Notes: ① when integrating over a vector field, orientation of C matters!



orientation of C matters!

→ (2) $\displaystyle\int_{-C} \vec{F}\cdot d\vec{r} = -\int_{C} F\cdot d\vec{r}$

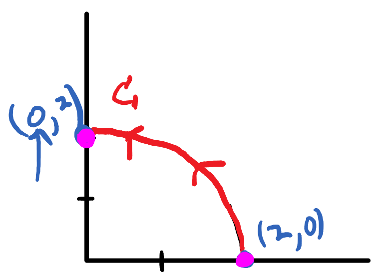


$\displaystyle\int_{C} \vec{F}\cdot(-\vec{T})\,ds$

(ex) compute the line integral of the vector field $\vec{F} = \langle 4x, 3y\rangle$ over the quarter circle part of $\boxed{x^2+y^2=4}$ ← $r=2$ (oriented counter-clockwise) in the first quadrant.

$$\int_C \vec{F}\cdot d\vec{r} = \underbrace{\boxed{\int_a^b \boxed{\vec{F}\cdot\vec{r}\,'(t)}\,dt}}$$

in terms of t.



$\boxed{\cos^2 t + \sin^2 t = 1}$

Parameterize $\boxed{x^2+y^2=4}$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$\frac{x}{2} = \cos t$, $\frac{y}{2} = \sin t$

→ $x = 2\cos t$, $y = 2\sin t$, $\boxed{0 \le t \le \frac{\pi}{2}}$

→ $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t\rangle$

$\vec{F}\cdot\vec{r}\,' = \langle 4x, 3y\rangle \cdot \langle -2\sin t, 2\cos t\rangle$

$$\vec{F} \cdot \vec{r}\,' = \langle 4x, 3y \rangle \cdot \langle -2\sin t, 2\cos t \rangle$$

$$= -8\underset{2\cos t}{x}\sin t + 6\underset{2\sin t}{y}\cos t$$

$$= -16\sin t\cos t + 12\sin t\cos t$$

$$= -4\sin t\cos t$$

$$\int_a^b \vec{F} \cdot \vec{r}\,' \, dt = -4\int_0^{\frac{\pi}{2}} \underbrace{\sin t}_{u}\underbrace{\cos t\, dt}_{du}$$

$$= -4\left[\frac{\sin^2 t}{2}\right]_0^{\frac{\pi}{2}}$$

$$= -2[1 - 0]$$

$$= -2$$

$$\vec{F} = \langle P, Q, R \rangle, \quad d\vec{r} = \langle dx, dy, dz \rangle$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r}} = \int_C \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle$$

$$= \boxed{\int_C P\,dx + Q\,dy + R\,dz}$$

Differential Form

ex Evaluate $\boxed{\int_C x^2\,dx + y^2\,dy + z^2\,dz}$ on the ex Evaluate $\boxed{\int_C x^2\,dx + y^2\,dy + z^2\,dz}$ on the line from $P(1,2,-1)$ to $Q(3,2,0)$

$x_0 \quad y_0 \quad z_0$

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \text{ where } \begin{pmatrix}\text{dir.}\\ \text{vec}\end{pmatrix} = \langle a, b, c\rangle$$

dir. vec.

$$\vec{PQ} = \langle 3-1,\ 2-2,\ 0-(-1)\rangle = \langle 2, 0, 1\rangle$$

$$\boxed{x = 1+2t,\quad y = 2,\quad z = -1+1t},\qquad 0 \le t \le 1$$

$dx = 2\,dt$, $dy = 0\,dt = 0$, $dz = dt$

$$\int x^2\,dx + \cancel{y^2\,dy} + z^2\,dz$$

0

$$\int_0^1 (1+2t)^2\,2\,dt + (-1+t)^2\,dt$$

$$\int_0^1 \left[(1+2t)^2\,2 + (-1+t)^2\right]dt$$

$$\int_0^1 \left[\underbrace{(1+4t+4t^2)\,2}_{2+8t+8t^2} + 1-2t+t^2\right]dt$$

$$= \int_0^1 (9t^2+6t+3)\,dt$$

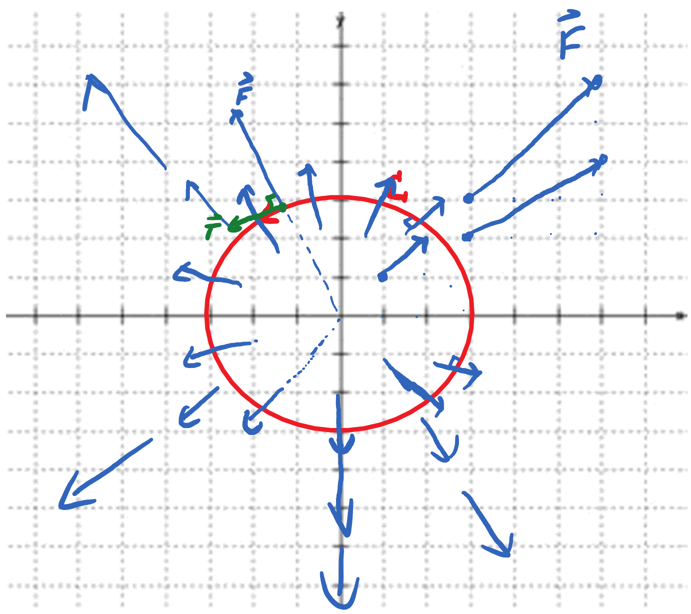
$$= \left[3t^3+3t^2+3t\right]_0^1$$

$$= 3+3+3$$

$$= 9$$

ex Find $\boxed{\int_C \vec{F}\cdot d\vec{r}}$ visually

a)



$\boxed{\vec{F} = x\,\vec{i} + y\,\vec{j}}$

Is $\int_C \vec{F}\cdot d\vec{r}$

< 0

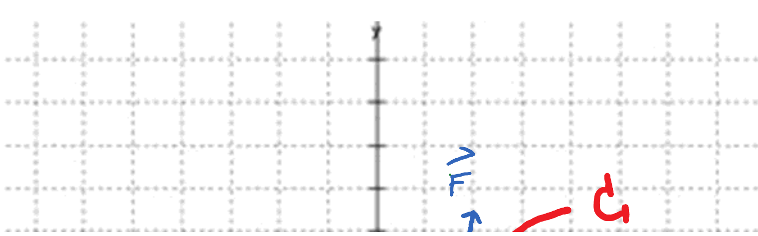
> 0

$= 0$?

$\int_C \vec{F}\cdot d\vec{r} = \int_C \underbrace{\vec{F}\cdot\vec{T}}_{0}\, d\vec{s}$

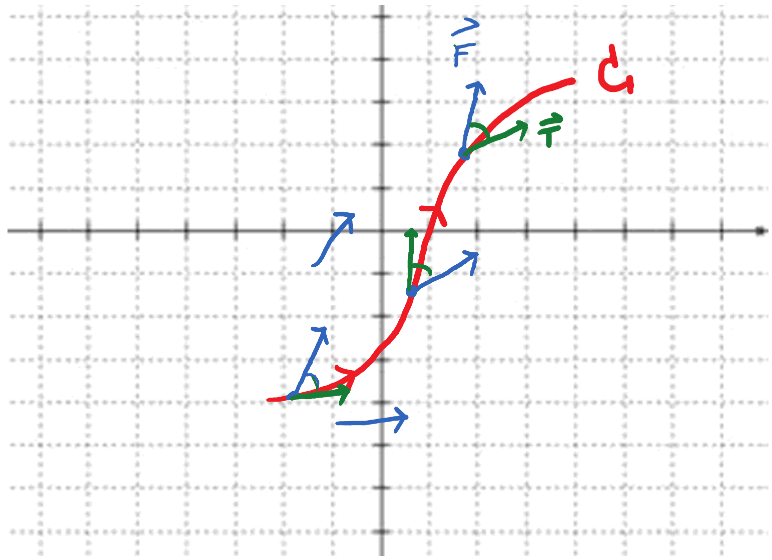
$= 0$

b)



$\vec{F}$ in blue.

Is $\int \vec{F}\cdot d\vec{r}$



Is $\int_C \vec{F}\cdot d\vec{r}$

< 0

> 0

$= 0$?

$\int_C \vec{F}\cdot\vec{T}\,ds$

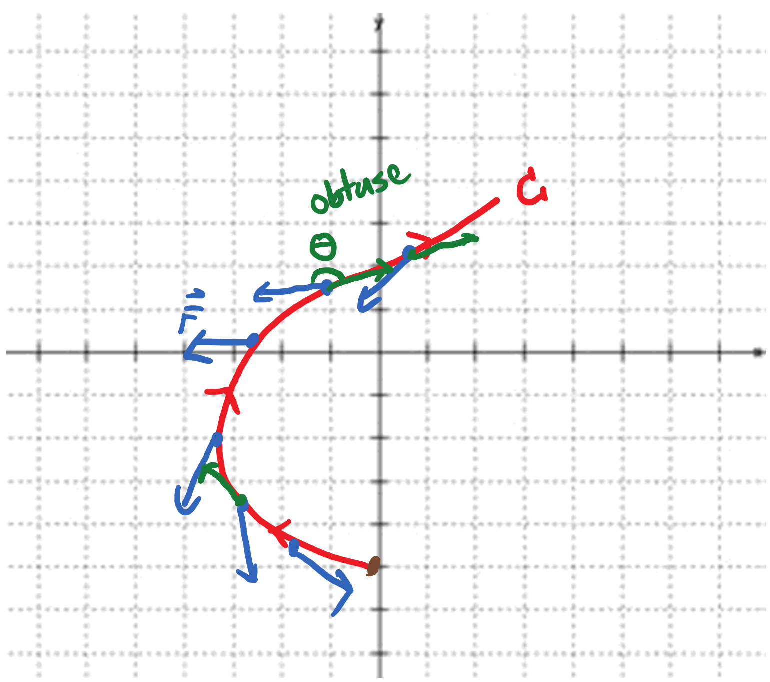
acute

$\vec{F}\cdot\vec{T} = \underbrace{\|\vec{F}\|\|\vec{T}\|}_{>0}\underbrace{\cos\theta}_{>0}$

obtuse

$\vec{F}\cdot\vec{T} = \underbrace{\|\vec{F}\|\ \|\vec{T}\|}_{>0}\underbrace{\cos\theta}_{<0} < 0$

c)



$\int_C \vec{F}\cdot d\vec{r} < 0$