Section 15.2: Iterated Integrals

Thursday, March 12, 2015 6:02 PM

Goals:

1. To evaluate an iterated integral.
2. To use an iterated integral to find the area of a rectangular plane region.
3. To evaluate a double integral over a rectangular region using an iterated integral.

ex Integrate $\int_1^2 (x^2 + 3y^2)\, dy$

$$= \left[x^2 y + y^3\right|_{y=1}^{y=2}$$

$$= (2x^2+8) - (x^2+1)$$

$$= x^2 + 7$$

ex Evaluate $\int_0^3 \underbrace{\left[\int_1^2 (x^2+3y^2)\, dy\right]} dx = 30$

$$\int_0^3 (x^2+7)\, dx$$

$$\left[\frac{x^3}{3} + 7x\right|_0^3$$

$$= \left(\frac{3^3}{3} + 7(3)\right) - 0$$

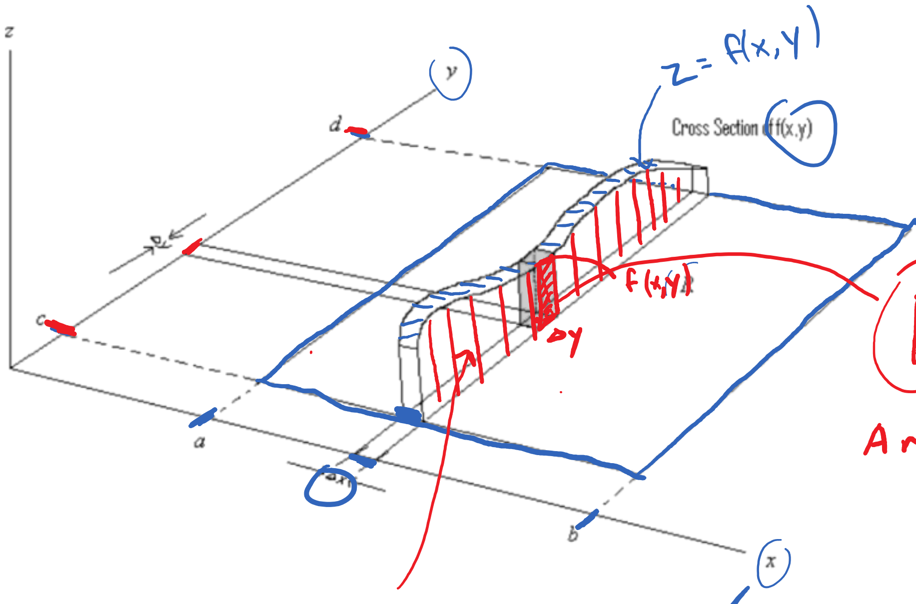
$= 9 + 21$

$= \boxed{30}$

Fubini's Theorem: Let f be continuous on $[a,b] \times [c,d]$. Then ...

$$\iint_R f(x,y)\, dA = \underbrace{\int_a^b \int_c^d f(x,y)\, dy\, dx = \boxed{\int_c^d \int_a^b f(x,y)\, dx\, dy}}$$

This says you get the same answer, no matter the order of integration.



slice

$\square$ $A = f(x,y)\,\Delta y$

Area of slice $= \int_c^d f(x,y)\, dy$

$$\begin{pmatrix}\text{Volume of slice} \\ \text{(with depth } \Delta x)\end{pmatrix} = \left[\int_c^d f(x,y)\, dy\right] \Delta x$$

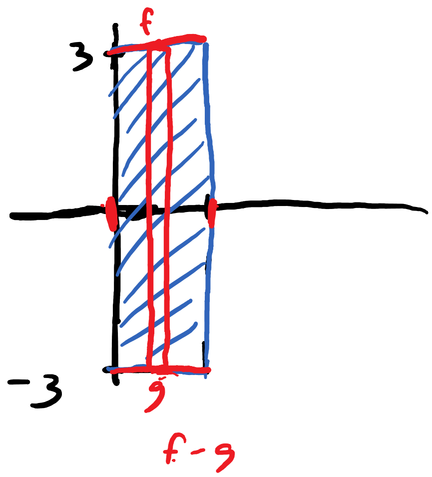
$$V = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

$$= \iint_R f(x,y)\, dA$$

(ex) Evaluate the double integral

$$\iint_R \frac{xy^2}{x^2+1}\, dA \quad , \quad R = \{(x,y) \mid 0 \le x \le 1,\ -3 \le y \le 3\}$$

$[0,1] \times [-3,3]$



$$\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx$$

$$\int_0^1 \int_{-3}^3 \left(\frac{x}{x^2+1}\right) y^2\, dy\, dx$$

$u = x^2+1$

$du = 2x\,dx$

$$= \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx \int_{-3}^3 y^2\, dy$$

$$= \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\, \underbrace{2x\,dx}_{du} \qquad \frac{1}{3}\left[y^3\right]_{-3}^3$$

$\frac{1}{u}\, du$

$$= \frac{1}{2}\left[\ln(x^2+1)\right]_0^1 \ \frac{1}{3}\left[y^3\right]_{-3}^3$$

$$= \frac{1}{2}\ln 2 \cdot \frac{1}{3}\left[27 - (-3)^3\right]$$

$$= \frac{1}{2}\ln 2 \cdot \frac{1}{3}[54]$$

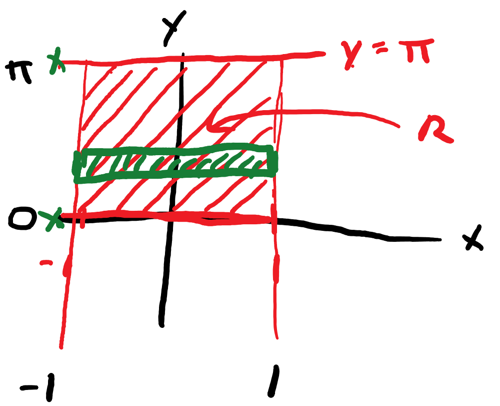
$$= 9\ln 2$$

Note: $\iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$

where $R = [a,b] \times [c,d]$

ex) Find the volume of the solid enclosed by $z = 1 + e^x \sin y$ and planes $\underbrace{x = \pm 1,\ y = 0,\ y = \pi}_{\text{lines in } xy\text{-plane}}$, $z = 0$

xy-plane



$$\int_0^\pi \left[\int_{-1}^{1} (1 + e^x \sin y)\,dx \right] dy$$

$$= \int_0^\pi \left[x + e^x \sin y \right]_{x=-1}^{x=1} dy$$

$$= \int_0^\pi \left[(1 + e\sin y) - (-1 + e^{-1}\sin y) \right] dy$$

$$= \int_0^\pi (2 + e\sin y - e^{-1}\sin y)\,dy$$

$$= \left[2y - e\cos y + e^{-1}\cos y \right]_0^\pi$$

$$= 2\pi - e(-1) + e^{-1}(-1) - (0 - e + e^{-1})$$

$$= 2\pi + 2e - \frac{2}{}$$

$$= 2\pi + 2e - \frac{2}{e}$$