Math 141
Sample Test ~~3~~ 2
Chamberlin

Name ________________________

In order to get all the points available on each problem, show how you arrive at your solutions. If an answer is correct even though your method is incorrect, then you will not get full credit.

1. Consider the region in Quadrant I bounded by $y = x^3$ and $y = 4x$. Set up the appropriate integrals for finding the volumes of revolution using the specified method and rotating about the specified axis. Be sure to draw the *i*th representative rectangle for each problem (**Note**: SET UP THE INTEGRALS ONLY. DO NOT EVALUATE)

a) Disc method about the *x*-axis
(**Hint**: $V_i = (\text{V}_{\text{big disc}}) - (\text{V}_{\text{little disc}})$)

$$V = \pi \int_0^2 (16x^2 - x^6)\,dx$$

b) Shell method about the *x*-axis

$$V = 2\pi \int_0^8 y\left(\sqrt[3]{y} - \frac{y}{4}\right) dy$$

c) Disc method about the line $x = -3$
(**Hint**: $V_i = (\text{V}_{\text{big disc}}) - (\text{V}_{\text{little disc}})$)

$$V = \pi \int_0^8 \left[\left(\sqrt[3]{y} + 3\right)^2 - \left(\frac{y}{4} + 3\right)^2\right] dy$$

d) Shell method about the line $x = 4$

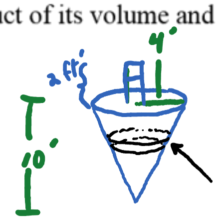
$$2\pi \int_0^2 (4 - x)(4x - x^3)\,dx$$

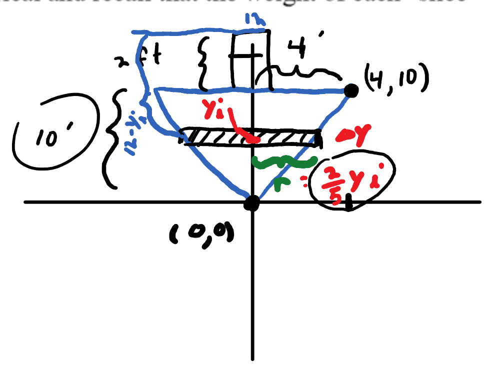
2. Set up the integral to calculate the area bounded by $y=\sqrt{x}$, $y=-2+x$, and the y-axis using vertical representative rectangles. (**Note**: SET UP THE INTEGRAL ONLY. DO NOT EVLUATE)

$$A = \int_0^4 \left(\sqrt{x} - (-2+x)\right)\,dx$$

$$= \int_0^4 \left(\sqrt{x} - x + 2\right)\,dx$$

3. Find the amount of work done in pumping all of the water out of a cone and up 2 feet above the cone if the top radius is 4 feet and height is 10 feet. Use the fact that water weighs 62.5 lb/ft^3. (**Hint**: Note that each "slice" of water is cylindrical and recall that the weight of each "slice" of water is the product of its volume and 62.5)





$y = mx+b$

$y = \frac{5}{2}x$

or

$x = \frac{2}{5}y$

$$W_i = F_i \cdot D_i$$

$$= (\text{wt of ith slice})\, D_i$$

$$= \left((62.5)(\text{vol of ith slice})\right) D_i$$

$$= 62.5\,(\pi r^2 h)\, D_i$$

$$= \underbrace{\left[62.5\,\pi\left(\tfrac{2}{5}y_i\right)^2 \Delta y\right]}_{\text{wt of ith slice}}(12-y_i)$$

$$W = 10\pi\int_0^{10} y^2(12-y)\,dy = 15000\pi \text{ ft}\cdot\text{lb}$$

4. A force of 16 lb is necessary to stretch a spring 0.5 feet past its natural length. How much work is done in stretching the spring to 1.5 feet longer than its natural length? (**Hint**: use Hooke's law: $F = kx$, where x represents distance that the spring is stretched past its natural length)

$F = kx$

$16 = k\, 0.5$

$k = 32$

$F(x) = 32x$

$$W = \int_0^{1.5} \underbrace{32x}_{F}\, \underbrace{dx}_{D}$$

$\vdots$

$= 36$ ft·lb

5. Find the average value of $f(x) = xe^{-x^2}$ on [-1, 2].

$$\text{Ave} = \frac{1}{2-(-1)} \int_{-1}^{2} xe^{-x^2}\,dx$$

$$= \frac{1}{3} \int_{-1}^{2} xe^{-x^2}\,dx$$

$\vdots$ u-sub

$$= -\frac{1}{6}\left[e^{-4} - e^{-1}\right]$$

6. Find the arc length of the graph of $f(x)=\frac{2}{3}x^{3/2}+1$ on the interval [1, 3]. Please leave your answer in exact form using only fractions (i.e. no decimals allowed!).

$$\frac{2}{3}\left[8-2\sqrt{2}\right]$$

7. Set up an integral that represents the area of the surface of revolution generated by revolving the following curve about the y-axis: $f(x)=\sqrt[3]{x}+2$, $1\le x\le 8$. (**Note: set up the integral only**, DO NOT EVALUATE THE INTEGRAL!!! However, you should find any derivative required by the area formula and simplify the integrand)

$$S = 2\pi\int_1^8 t\sqrt{\frac{1}{9t^{4/3}}+1}\,dt$$

(assuming $x=t$
and $y=\sqrt[3]{t}+2$
for $1\le t\le 8$)

8. Consider the parametric equations $x = 2\sin t$, $y = 3\cos t$ where $0 \le t \le 2\pi$.

a) Eliminate the parameter to find a Cartesian equation of the curve.

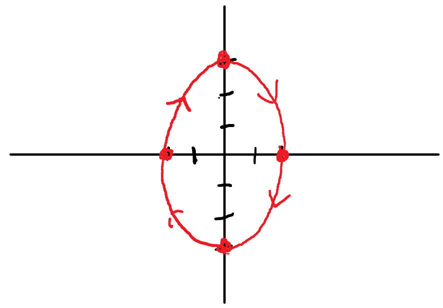
$\frac{x}{2} = \sin t$, $\frac{y}{3} = \cos t$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

b) Graph the curve.

ellipse

center: (0, 0)



c) describe the motion (and direction) of a particle with position (x, y) on the curve as t goes from 0 to 2π.

start point

t	x	y
0	0	3
$\frac{\pi}{2}$	2	0

goes clockwise

It goes clockwise once around the ellipse starting at $(0, 3)$.

9. a) Consider the parametric equations $x = t^3 - 3t^2$, $y = t^3 - 3t$. Find $\frac{dy}{dx}$ in terms of t.

$$\frac{dy}{dx} = \frac{t^2 - 1}{t^2 - 2t}$$

b) Find the points where the tangent line is vertical.

$(0,0)$ $(-4,2)$

10. Use the parametric formula to find the area of the surface of revolution generated by revolving the following curve about the y-axis: $x = t$, $y = 2t$ where $0 \le t \le 4$.

$16\pi\sqrt{5}$

11. Convert to a Cartesian equation and identify the curve by writing it in standard form: $r = 3\cos\theta$

$$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$$

Circle w/ center $\left(\frac{3}{2}, 0\right)$

and radius $= \frac{3}{2}$

12. Find the points of horizontal tangency to the polar curve $r = 1 + \sin\theta$

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$x = (1+\sin\theta)\cos\theta$ | $y = (1+\sin\theta)\sin\theta$

$\boxed{x = \cos\theta + \sin\theta\cos\theta \quad y = \sin\theta + \sin^2\theta}$

parametric eqn

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$\boxed{\text{H-tan}}$ set $\frac{dy}{d\theta} = 0$

$$\frac{dy}{d\theta} = \boxed{\cos\theta + 2\sin\theta\cos\theta = 0}$$

$$\cos\theta(1 + 2\sin\theta) = 0$$

$\cos\theta = 0$ or $1 + 2\sin\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $\sin\theta = -\frac{1}{2}$ $\quad$ $\left(\alpha = \frac{\pi}{6}\right)$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\frac{3\pi}{2}$

$r = 1 + \sin\theta$

$r = 1 + \sin\theta$ $\frac{3\pi}{2}$

$\boxed{\left(2, \frac{\pi}{2}\right) \quad \cancel{\left(0, \frac{3\pi}{2}\right)} \quad \left(\frac{1}{2}, \frac{7\pi}{6}\right) \quad \left(\frac{1}{2}, \frac{11\pi}{6}\right)}$ Answer

$x = \cos\theta + \sin\theta\cos\theta$

$$\frac{dx}{d\theta} = \boxed{-\sin\theta + \underbrace{\cos^2\theta}_{1-\sin^2\theta} - \sin^2\theta = 0}$$

$$-\sin\theta + 1 - 2\sin^2\theta = 0$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$\sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = -1$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \theta = \frac{3\pi}{2}$$

has a vertical tangent at $\left(0, \frac{3\pi}{2}\right)$ (see graph)

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{3\pi}{2}} = \frac{0}{0} \quad \text{problem}$$

13. Find the area enlcosed by the inner loop of $r = 1 + 2\cos\theta$.