Math 141 Name Key

Sample Questions Test 3

In order to get all the points available on each problem, show how you arrive at your solutions. If an answer is correct even though your method is incorrect, then you will not get full credit. Show all relevant details in the space provided (no separate sheets of scratch paper please).

1. a) Write a formula for the nth term of the sequence:

$$\frac{1}{3}, \frac{8}{5}, \frac{27}{7}, \frac{64}{9}, \frac{125}{11}, \ldots$$

$$a_n = \frac{n^3}{2n+1}$$

b) Find the limit of $a_n = \dfrac{n}{2\sqrt{n^2+1}}$

$$\lim_{n\to\infty} \frac{n}{2\sqrt{n^2+1}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{2\sqrt{1+\cancelto{0}{\frac{1}{n^2}}}} = \frac{1}{2}$$

2. Determine whether or not the following series converges absolutely, conditionally or if it diverges. State reasons for each of your answers: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n+1}}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+1}}$$

Let $b_n = \dfrac{1}{\sqrt[3]{n}} = \dfrac{1}{n^{1/3}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n+1}} \to 1 > 0.$$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges ($p = \frac{1}{3} < 1$). So, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+1}}$ diverges by LCT.

Thus, the given series does not converge absolutely.

Next, consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n+1}}$ as an alternating series.

Let $b_n = \dfrac{1}{\sqrt[3]{n+1}} > 0$.

① $\dfrac{1}{\sqrt[3]{2}} > \dfrac{1}{\sqrt[3]{3}} > \dfrac{1}{\sqrt[3]{4}} > \ldots$

So, b_n decreases

② $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt[3]{n+1}} = 0$

So, the series converges by AST.

Thus, the series converges conditionally.

3. Determine whether the series converges or diverges.

a) $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^5}$

$$0 \le \frac{|\sin(n)|}{n^5} \le \frac{1}{n^5}.$$

Now, $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges ($p = 5 > 1$). So $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^5}$ converges by Comparison.

b) $\sum_{n=0}^{\infty} \left(\frac{7n+5}{2n+3}\right)^n$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{7n+5}{2n+3}\right)^n} = \frac{7n+5}{2n+3}$$

$$\lim_{n\to\infty} \frac{7n+5}{2n+3} = \frac{7}{2} > 1.$$

Thus, $\sum_{n=0}^{\infty} \left(\frac{7n+5}{2n+3}\right)^n$ diverges by the Root Test.

4. Find the sum of the convergent series: $\sum_{n=1}^{\infty} \left(\frac{1}{4^n} + \frac{1}{3^n}\right)$

$$= \sum_{n=1}^{\infty} \frac{1}{4^n} + \sum_{n=1}^{\infty} \frac{1}{3^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

Note that these are both geometric series with common ratios $\frac{1}{4}$ and $\frac{1}{3}$ that are between -1 and 1.

$$= \frac{\frac{1}{4}}{1-\frac{1}{4}} + \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} + \frac{\frac{1}{3}}{\frac{2}{3}}$$

$$= \frac{1}{3} + \frac{1}{2}$$

$$= \frac{5}{6}$$

5. Use the Integral Test to determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

Let $f(x) = \frac{x}{x^2+3}$.

① $f(x) > 0$ for $x \geq 1$

② $f(x)$ is a continuous rational function for $x \geq 1$.

③ $f'(x) = \frac{(x^2+3)\cdot 1 - x(2x)}{(x^2+3)^2}$

$$= \frac{x^2+3-2x^2}{(x^2+3)^2}$$

$$= \frac{-x^2+3}{(x^2+3)^2}.$$

Note that $-x^2+3<0$ when $x^2>3$ or $x>\underset{\approx 1.7}{\sqrt{3}}$ (or when $x<-\sqrt{3}$, which isn't relevant here).

So, the above derivative is negative as long as $x>2$. Thus $f(x)$ decreases when $x>2$.

$$\int_1^\infty \frac{x}{x^2+3}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2}\int_1^b \frac{2x}{x^2+3}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\ln(x^2+3)\Big]_1^b$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\ln(\overset{\infty}{\cancel{b^2}}+3) - \ln(4)\Big]$$

$$= \infty$$

So, $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$ diverges by I.T.

6. Approximate the sum of the convergent series with an error less than 0.001: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

$$b_n = \frac{1}{(2n)!}$$

$$b_4 = \frac{1}{8!} \approx 0.000024 < 0.001$$

So, by ASET, we'll take S_3 as our estimate of the sum.

$$S_3 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!}$$

4 terms since n starts at 0.

$$\approx 0.54$$

7. Find the interval of convergence **and** the radius of convergence of the power series (don't forget to check for convergence at the endpoints of your IOC): $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right|$$

$$= \left|x \cdot \frac{n}{n+1}\right|$$

$$= |x| \frac{n}{n+1}$$

Next, $\lim_{n\to\infty} |x| \frac{n}{n+1}$

$$= |x| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= |x| \cdot 1$$

$$= |x|$$

So, by the Ratio Test, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges when $|x| < 1$.

End points:

$\boxed{x=-1}$ gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Let $b_n = \frac{1}{n} > 0$.

① b_n decreases

② $\lim_{n\to\infty} \frac{1}{n} = 0$

So, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by AST.

$\boxed{x=1}$ gives $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges ($p = 1 \le 1$).

Thus $I = [-1, 1)$ and $R = 1$