Section 8.3 Continued

Friday, March 14, 2014
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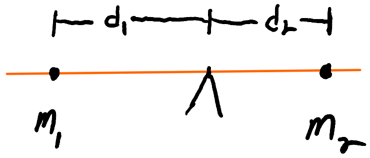
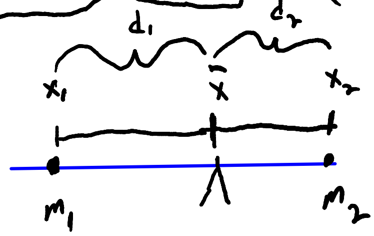
Goal: To find the center of mass of a planar lamina (i.e. a thin flat sheet of some material).

* see saw



balances if $\boxed{m_1 d_1 = m_2 d_2}$ (Lever principle)



$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

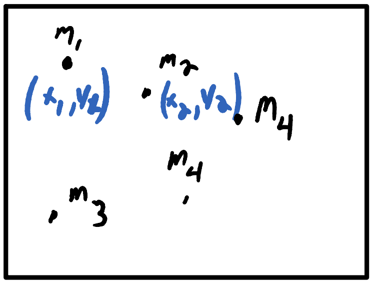
Now, say we have n point masses, $m_1, m_2 \ldots m_n$ at points $x_1, x_2, \ldots, x_n$, then

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

m

$$\bar{x} = \frac{1}{m} \sum_{i=1}^{n} m_i x_i \quad *$$

If you have point masses in plane



balance point (or centroid) is ...

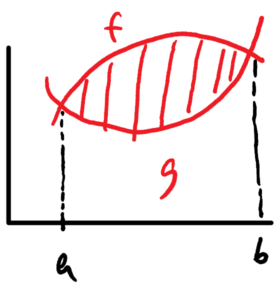
$$\underbrace{(\bar{x},\bar{y})}_{\text{centroid}} = \left(\frac{1}{m}\sum_{i=1}^{n} m_i x_i \, , \, \frac{1}{m}\sum_{i=1}^{n} m_i y_i\right) \quad ★★$$

or center of mass

or balance point

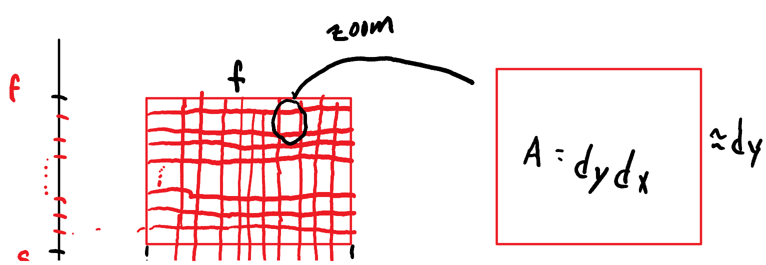
$\boxed{\text{Time-out}}$

$$\int_a^b \int_{g(x)}^{f(x)} dy\,dx = \text{area of} \ldots$$

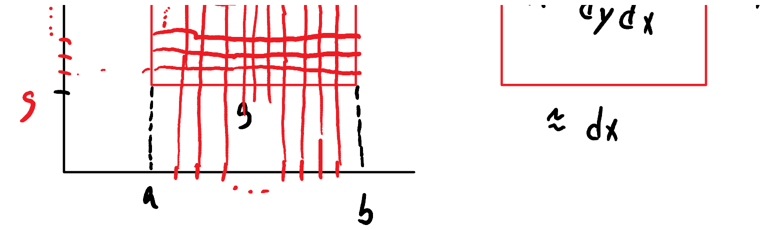


suppose f and g are constant (temporarily)

$$\int_a^b \overbrace{\left[\int_g^f dy\right]}^{(f-g)} dx$$



a [g]



$$\int_a^b \int_g^f \underbrace{dy\,dx}_{\text{area element}}$$

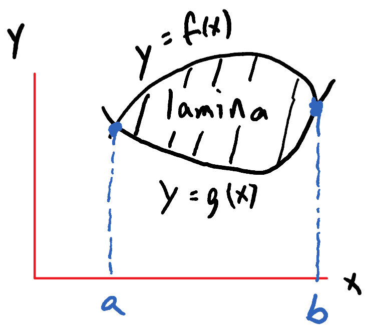
area element → even if f and g aren't constant

Think of this integral as summing all area elements to give total area in xy-plane.

$\boxed{\text{Time-in}}$

Def (1) A <u>planar lamina</u> is a flat sheet of some material

flat sheet: corresponds to a region in xy-plane



rho

(2) The density at any spot on the lamina is ρ and ρ is measured in $\frac{\text{Mass}}{\text{area}}$.

ρ constant

A = area of lamina

(3) $$m = \begin{pmatrix}\text{mass of} \\ \text{lamina}\end{pmatrix} = \rho A = \rho \underbrace{\int_a^b \int_{g(x)}^{f(x)} dy\,dx}_{A}$$

b $f(x)$ } Think of this

$$= \int_a^b \int_{g(x)}^{f(x)} \underbrace{\rho \, dy\, dx}_{\text{mass element}}$$

A

Think of this integral as summing all such mass elements to give total mass of the lamina

4. By analogy with ☆☆

time-capsule for calc III

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{m} \int_a^b \int_{g(x)}^{f(x)} x \rho \, dy\, dx \\ \bar{y} &= \frac{1}{m} \int_a^b \int_{g(x)}^{f(x)} y \rho \, dy\, dx \end{aligned}}$$

$(\bar{x}, \bar{y})$ gives the center of mass for planar lamina (a.k.a the centroid)

simplify by pulling ρ outside integrals

$$\left(\frac{\rho}{m} = \frac{\rho}{\rho A} = \frac{1}{A} \right)$$

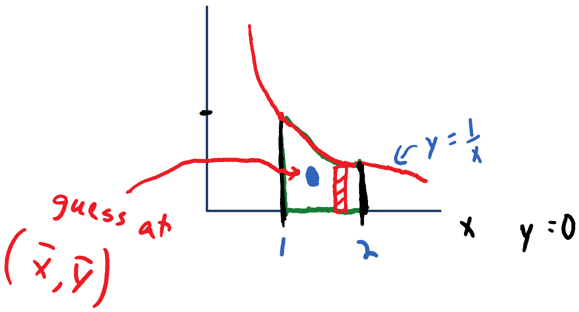
important

$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{1}{A} \int_a^b \int_{g(x)}^{f(x)} x \, dy\, dx, \ \frac{1}{A} \int_a^b \int_{g(x)}^{f(x)} y \, dy\, dx \right)}$$

(ex) Find the centroid

a) $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 2$

y



$$A = \int_1^2 \left[\int_0^{\frac{1}{x}} 1\,dy\right] dx$$

$$= \int_1^2 [y]_0^{\frac{1}{x}}\,dx$$

$$= \int_1^2 \left(\frac{1}{x} - 0\right) dx$$

$$= \int_1^2 \frac{1}{x}\,dx$$

$$= [\ln x]_1^2$$

$$A = \ln 2$$

$$\bar{x} = \frac{1}{\ln 2} \int_1^2 \left[\int_0^{\frac{1}{x}} x\,dy\right] dx$$

$$= \frac{1}{\ln 2} \int_1^2 x[y]_0^{\frac{1}{x}}\,dx$$

$$= \frac{1}{\ln 2} \int_1^2 x\left(\frac{1}{x}\right) dx$$

$$= \frac{1}{\ln 2} \int_1^2 dx$$

$$= \frac{1}{\ln 2} [x]_1^2$$

$$= \frac{1}{\ln 2}(2-1)$$

$$\bar{x} = \frac{1}{\ln 2}$$

$$\bar{y} = \frac{1}{\ln 2} \int_1^2 \int_0^{\frac{1}{x}} y\,dy\,dx$$

$$= \frac{1}{2} \cdot \frac{1}{\ln 2} \int_1^2 [y^2]_0^{\frac{1}{x}}\,dx$$

$$= \frac{1}{2} \cdot \frac{1}{\ln 2} \int_{1} \left[y^2\right]_0^{\frac{1}{x}} dx$$

$$= \frac{1}{2\ln 2} \int_1^2 \left(\frac{1}{x^2} - 0\right) dx$$

$$= \frac{1}{2\ln 2} \int_1^2 x^{-2} \, dx$$

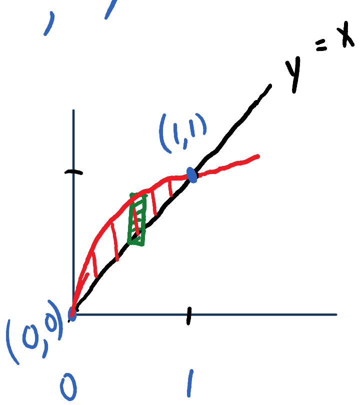
$$= -\frac{1}{2\ln 2} \left[\frac{1}{x}\right]_1^2$$

$$= -\frac{1}{2\ln 2}\left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{4\ln 2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{\ln 2}, \frac{1}{4\ln 2}\right)$$

b) $y = \sqrt{x}$, $y = x$



$$A = \int^1 \int^{\sqrt{x}} dy\,dx = \frac{1}{6}$$

$$A = \int_0^1 \int_x^1 dy\,dx = \frac{1}{6}$$

$$\bar{x} = 6 \int_0^1 \int_x^{\sqrt{x}} x \, dy\, dx = \frac{2}{5}$$

$$\bar{y} = 6 \int_0^1 \int_x^{\sqrt{x}} y \, dy\, dx = \frac{1}{2}$$

So, $(\bar{x}, \bar{y}) = \left(\frac{2}{5}, \frac{1}{2}\right)$