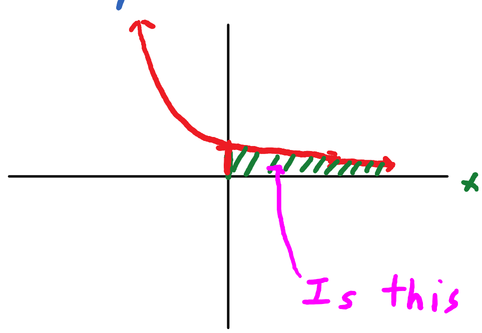
Improper Integrals

Goal: To evaluate integrals with either infinite limits of integration or an integrand with an infinite discontinuity in the interval of integration.

ex Evaluate

a) $\displaystyle\int_0^\infty e^{-x}\,dx$



$$\overset{\text{def}}{=} \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} -\left[e^{-x}\right]_0^b$$

$$= \lim_{b\to\infty} -\left[\overset{0}{e^{-b}} - 1\right]$$

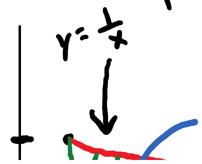
$$= 1$$

integral converges

yes area is finite

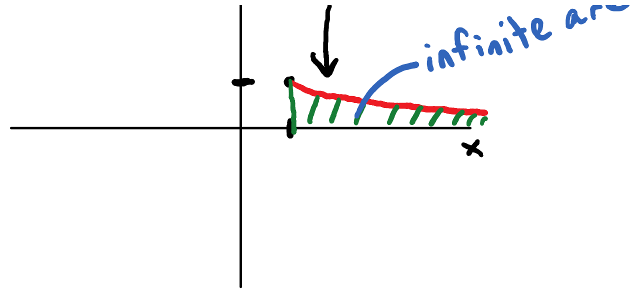
When an improper integral equals a finite number, we say it converges. Otherwise, we say it diverges.

b) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$



$$\lim \int^b \frac{1}{x}\,dx$$

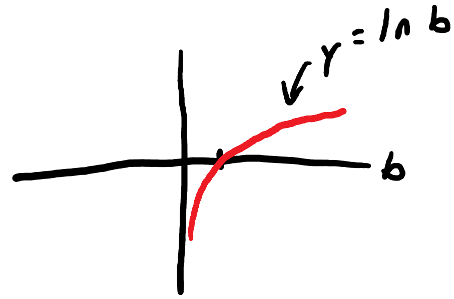
$$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$



$$\lim_{b\to\infty} \Big[\ln x\Big]_1^b$$

$$\lim_{b\to\infty} \Big[\underbrace{\ln b}_{\infty} - \overbrace{\ln 1}^{0}\Big]$$

$$= \infty$$



So the integral diverges

c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ (integrand is even, so you could evaluate just one of the integrals and muptiply by 2).

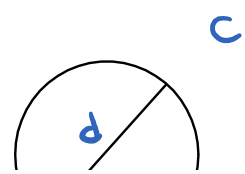
$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx$$

$$\lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$\lim_{a\to-\infty} \Big[\arctan x\Big]_a^0 + \lim_{b\to\infty} \Big[\arctan x\Big]_0^b$$

$$\lim_{a\to-\infty} \Big[\overbrace{\arctan 0}^{0} - \arctan a\Big] + \lim_{b\to\infty} \Big[\underbrace{\arctan b}_{\frac{\pi}{2}} - \overbrace{\arctan 0}^{0}\Big]$$

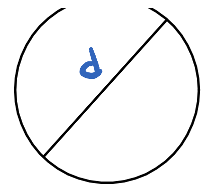
$$0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$



$0 - \left(-\frac{\pi}{2}\right) \quad + \quad \frac{\pi}{2}$

$\frac{\pi}{2} + \frac{\pi}{2}$

π



$\pi = \frac{C}{d}$

d) $\int_1^{\infty} (1-x)\, e^{-x}\, dx$

Time-out

$\int u\,dv = uv - \int v\,du$

$$\int \underbrace{(1-x)}_{u}\underbrace{e^{-x}\,dx}_{dv}$$

$u = 1 - x \qquad dv = e^{-x}\,dx$

$du = -\,dx \qquad v = -e^{-x}$

$= -e^{-x}(1-x) - \int -e^{-x}(-1)\,dx$

$= -e^{-x} + xe^{-x} - \int e^{-x}\,dx$

$= \cancel{-e^{-x}} + xe^{-x} - (\cancel{-e^{-x}}) + C$

$= xe^{-x} + C$

Time-in

$\int_1^{\infty} (1-x)\, e^{-x}\, dx$

$\lim_{b\to\infty} \int_1^{b} (1-x)\, e^{-x}\, dx$

$\left[\ {}^{1}e^{-1} \quad \right]^{b}$

$b \to \infty \qquad 1$

$$\lim_{b\to\infty} \left[x e^{-x}\right]_1^b \qquad 1e^{-1}$$

$$\lim_{b\to\infty} \left[b e^{-b} - \frac{1}{e}\right]$$

$$\lim_{b\to\infty} \left[b e^{-b}\right] - \frac{1}{e} \qquad \infty \cdot 0$$

$$\lim_{b\to\infty} \frac{b}{e^b} - \frac{1}{e} \qquad \frac{\infty}{\infty} \checkmark$$

$$\overset{H}{=} \left[\lim_{b\to\infty} \frac{1}{e^b}\right] - \frac{1}{e} \qquad 0$$

$$= -\frac{1}{e}$$

ex) Improper integrals with infinite discontinuities

a) $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$

b) $\displaystyle\int_{-1}^{2} \frac{1}{x^3}\,dx$

$$\overset{\text{def}}{=} \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_{0}^{2} \frac{1}{x^3}\,dx$$

both must converge to say the original integral converges

$$= \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^3}\,dx \quad + \;\ldots$$

$$\int \frac{1}{x^3}\,dx$$

$$\int x^{-3}\,dx$$

$$-\frac{1}{2}x^{-2}$$

$$-\frac{1}{2x^2}$$

$$= \lim_{b\to 0^-} -\left[\frac{1}{2x^2}\right]_{-1}^{b}$$

$$= \lim_{b\to 0^-} -\left[\overset{\infty}{\cancel{\frac{1}{2b^2}}} - \frac{1}{2}\right]$$

$$= -\infty$$

(don't bother evaluating the other limit)

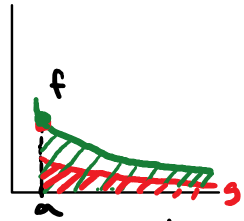
So, the integral diverges

Comparison Theorem

Comparison Theorem

f, g continuous with $0 \le g(x) \le f(x)$.

For $x \ge a$.



a) If $\int_a^\infty f(x)\,dx$ converges then $\int_a^\infty g(x)\,dx$ converge

b) If $\int_a^\infty g(x)\,dx$ diverges then $\int_a^b f(x)\,dx$ diverges

theorem

$\boxed{\text{Note}}$: $\boxed{\int_1^\infty \frac{1}{x^p}\,dx}$ is convergent if $p > 1$

and divergent if $\boxed{p \le 1}$ ex 4

ex Does it converge?

a) $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$

① $0 \le \frac{x}{\sqrt{1+x^6}} \le \frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$

② Now, $\int_1^\infty \frac{1}{x^2}\,dx$ converges $(p = 2 > 1)$

③ So, $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ converges by Comparison

(3) So, $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ converges by Comparison

b) $\int_1^\infty \frac{2+e^{-x}}{x}\,dx$

(1) $0 \le \frac{1}{x} \le \frac{2}{x} \le \frac{2+e^{-x}}{x}$

(2) Now $\int_1^\infty \frac{1}{x}\,dx$ diverges ($p=1 \le 1$)

(3) So, the original integral diverges by comparison.

c) $\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

(1) $0 \le \frac{\arctan x}{2+e^x} \le \frac{\frac{\pi}{2}}{2+e^x} \le \frac{2}{2+e^x} \le \frac{2}{e^x}$

(2) Note that $\int_0^\infty \frac{2}{e^x}\,dx \overset{\text{evaluate}}{(\dots)} = 2$, which means this integral converges

(3) Thus, $\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$ converges by comparison