Section 7.4: Integration of Rational Functions

Friday, January 24, 2014 1:26 PM

Goals:

1. To integrate "improper" rational functions after performing long division.
2. To integrate rational functions after performing partial fraction decomposition

ex $$\int \frac{4x^2}{x^2+9}\,dx$$

when the degree of NUM ≥ degree of DEN, long divide to rewrite integrand in terms of simpler fctns that we can integrate

what I mean by "improper"

$$\begin{array}{r|l} & 4 \\ x^2+9 & 4x^2+0x+0 \\ & -4x^2 \quad +36 \\ \hline & -36 \end{array}$$

same

$$4-\frac{36}{x^2+9}$$

$$\int \left(4-\frac{36}{x^2+9}\right)dx$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}}$$

relevant integration formula

$$4x-36\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$$

$$4x-12\tan^{-1}\left(\frac{x}{3}\right)+C$$

ex Partial Fration Decomposition

a) Distinct Linear factors in the denominator

$$\frac{1}{x^2-5x+6}$$

$$\frac{\quad}{x^2-5x+6}$$

$$\left[\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}\right](x-2)(x-3)$$

$$1 = A(x-3) + B(x-2)$$

$\boxed{\text{Let } x=2}$

$$1 = A(-1) + 0$$

$$A = -1$$

$\boxed{\text{Let } x=3}$

$$1 = 0 + B$$

$$B = 1$$

$$\frac{1}{(x-2)(x-3)} = \boxed{\frac{-1}{x-2} + \frac{1}{x-3}}$$ decomposition

b) Repeated linear factors

$$\frac{5x^2+20x+6}{x^3+2x^2+x}$$

$$\left[\frac{5x^2+20x+6}{x(x+1)^2} = \boxed{\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}}\right]x(x+1)^2$$

$$5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$$

$\boxed{\text{Let } x=-1}$

$$5-20+6 = -C$$

$$C = 9$$

$\boxed{\text{Let } x=0}$

$$6 = A$$

$$A = 6$$

$\boxed{x=1}$

$$31 = 24 + 2B + 9$$

$$-2 = 2B$$

$$B = -1$$

$$\boxed{\frac{6}{x} + \frac{-1}{x+1} + \frac{9}{(x+1)^2}}$$

c) Repeated Quadratic Factors

$$\frac{8x^3+13x}{(x^2+2)^2}$$

$$\left[\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}\right] (x^2+2)^2$$

$$8x^3+13x = (Ax+B)(x^2+2) + Cx+D$$

$$8x^3+13x = Ax^3+2Ax+Bx^2+2B+Cx+D$$

$$8x^3+13x = Ax^3+Bx^2+\underbrace{2Ax+Cx}+2B+D$$

$$8x^3+13x = Ax^3+Bx^2+(2A+C)x+(2B+D)$$

$$A = 8$$
$$B = 0$$
$$2A+C = 13$$
$$2B+D = 0$$

set corresponding coefficients equal

$$16+C = 13$$

$$C = -3$$

$$D = 0$$

$$\frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

$$\frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2}$$

ex Evaluate

a) $\int \frac{1}{x^2-5x+6}\, dx$

$\vdots$ partial fraction

$$\int \left(\frac{-1}{x-2} + \frac{1}{x-3}\right) dx$$

$$= -\int \frac{1}{x-2}\, dx + \int \frac{1}{x-3}\, dx$$

$$= -\ln|x-2| + \ln|x-3| + c$$

b) $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\, dx$

$\vdots$ pfd

$$\int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right) dx$$

$$\int \frac{6}{x} dx - \int \frac{1}{x+1} dx + 9\int \frac{1}{(x+1)^2} dx$$

$$\underbrace{6\ln|x| - \ln|x+1|} - \frac{9}{x+1} + c$$

$$\ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + c$$

c) $\int \frac{8x^3+13x}{(x^2+2)^2}\, dx$

$\vdots$ pfd

$$\int \left(\frac{8x}{x^2+2} - \frac{3x}{(x^2+2)^2}\right) dx$$

$$4\int \frac{1}{x^2+2}\, 2x\, dx \; - \frac{3}{2}\int \frac{1}{(x^2+2)^2}\, 2x\, dx$$

$$4\int \frac{1}{x^2+2}\, 2x\,dx \quad -\frac{3}{2}\int \frac{1}{(x^2+2)^2}\, 2x\,dx$$

$$4\ln|x^2+2| + \frac{3}{2}\cdot (x^2+2)^{-1} + C$$

$$4\ln|x^2+2| + \frac{3}{2(x^2+2)} + C$$