Section 6.1: Areas Between Curves

Monday, February 10, 2014
11:41 AM

Goal: To use definite integrals to find the area bounded by two curves.

Def of a definite integral:

$$\int_a^b f(x)\,dx = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x}$$

when $f(x) \geq 0$

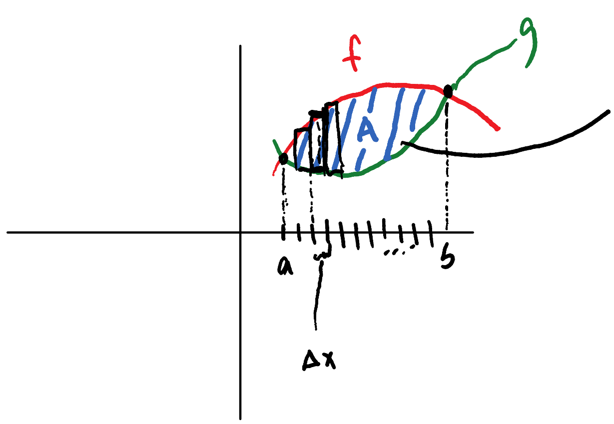
$$\int_a^b f(x)\,dx = \int_a^b (f(x) - \underbrace{0}_{g(x)=0})\,dx = \text{Area}$$

n subintervals

c_i is a sample x-value from the ith interval

Δx is the width of the ith subinterval

Area between two curves, $y = f(x)$ and $y = g(x)$



$$= \int_a^b [f(x) - g(x)]\,dx$$

Theorem: If functions f and g are continuous on [a,b] and g(x) is less than or equal to f(x) for all x in [a,b], then the area bound by the graphs of f and g and the line x = a and x = b is ...

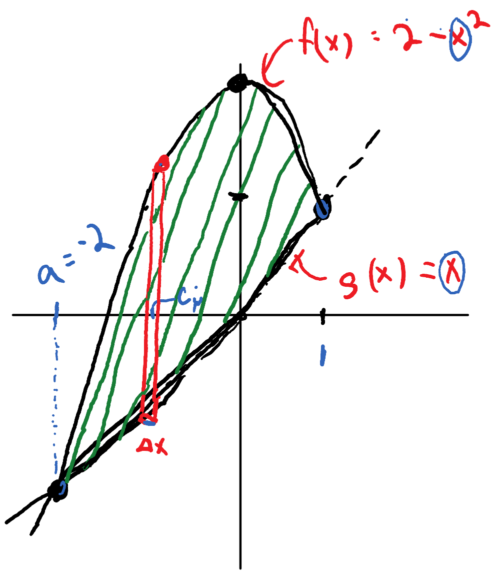
$$A = \int_a^b \left(f(x) - g(x)\right) dx$$ ← notation

these are equivalent

Note: In terms of the limit of a Riemann sum

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[f(c_i) - g(c_i)\right] \Delta x$$

ex Find the area bounded by $f(x) = 2 - x^2$ and $g(x) = x$



To find "a" set

$$2 - x^2 = x$$

$$0 = x^2 + x - 2$$

$$0 = (x - 1)(x + 2)$$

$$x = 1, \quad x = -2$$

Area of ith subrectangle: $A_i = \left[(2 - c_i^2) - c_i\right] \Delta x$

$$A = \int_a^b \left[f(x) - g(x)\right] dx = \int_{-2}^{1} \left[\overbrace{(2 - x^2)}^{f} - \overbrace{x}^{g}\right] dx$$

$$\int_{-2}^{1} \left[\int_{x}^{2-x^2} 1 \, dy \right] dx$$

$$= \int_{-2}^{1} \left(-x^2 - x + 2\right) dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{1}$$

$$= \left[\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} \overbrace{- 2 - 4}^{-6}\right)\right]$$

$$= \left(-\frac{2}{6} - \frac{3}{6} + \frac{12}{6}\right) - \left(\frac{8}{3} - \frac{18}{3}\right)$$

$$= \frac{7}{6} + \frac{10}{3}$$

$$= \frac{27}{6}$$

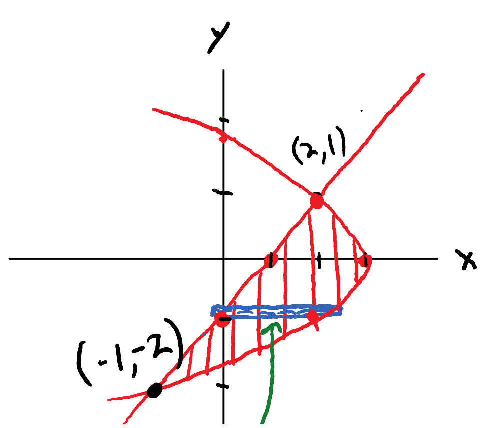
$$= \frac{9}{2}$$

$$\boxed{\int_{-2}^{1}\int_{x}^{2-x^2} dy\,dx}$$

Double Integral setup

ex Find the area between $\underbrace{x = 3 - y^2}_{x = f(y)}$ and $\underbrace{y = x - 1}_{\boxed{x = y + 1}}$

use horizontal representative rectangle and set up integral



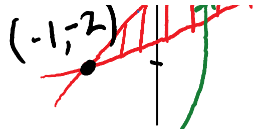
$$3 - y^2 = y + 1$$

$$0 = y^2 + y - 2$$

$$0 = (y - 1)(y + 2)$$

$$y = 1, \quad y = -2$$

rectangle set up integral in terms of y



representative rectangle

$y = 1, \quad y = -2$

area of representative rectangle $\left\{ A_i = \left[\left(3 - c_i^2\right) - \left(c_i + 1\right)\right] \Delta y \right.$

$$A = \int_{-2}^{1} \left[\left(3 - y^2\right) - \left(y+1\right)\right] dy$$

$$= \int_{-2}^{1} \left[-y^2 - y + 2\right] dy$$

$$= \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y\right]_{-2}^{1} = \frac{9}{2}$$

Every definite integral can be set up as a double integral as follows...

$$A = \int_a^b \left[f(x) - g(x)\right] dx = \int_a^b \left[\int_{g(x)}^{f(x)} 1\, dy\right] dx = \int_a^b \int_{g(x)}^{f(x)} dy\, dx,$$

since

$$\int_{g(x)}^{f(x)} 1\, dy$$

$$= \left[y\right]_{g(x)}^{f(x)}$$

$$= \left[f(x) - g(x)\right]$$