Goal: To evaluate limits using L'Hospital's Rule

Warm-up: Find ① $\lim\limits_{x \to \infty} \dfrac{3x^2-1}{2x^2+1}$ and ② $\lim\limits_{x \to -1} \dfrac{2x^2-2}{x+1}$

$$\lim_{x\to\infty} \frac{(3x^2-1)\frac{1}{x^2}}{(2x^2+1)\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{3\cancel{x^2}}{\cancel{x^2}} - \frac{1}{x^2}\nearrow 0}{\frac{2\cancel{x^2}}{\cancel{x^2}} + \frac{1}{x^2}\nearrow 0} = \frac{3}{2}$$

$$= \lim_{x\to\infty} \frac{6x}{4x} = \frac{3}{2}$$

L'Hospital's Rule (H)

Let f and g be differentiable and $g'(x) \neq 0$ near a (except maybe at a). Also let

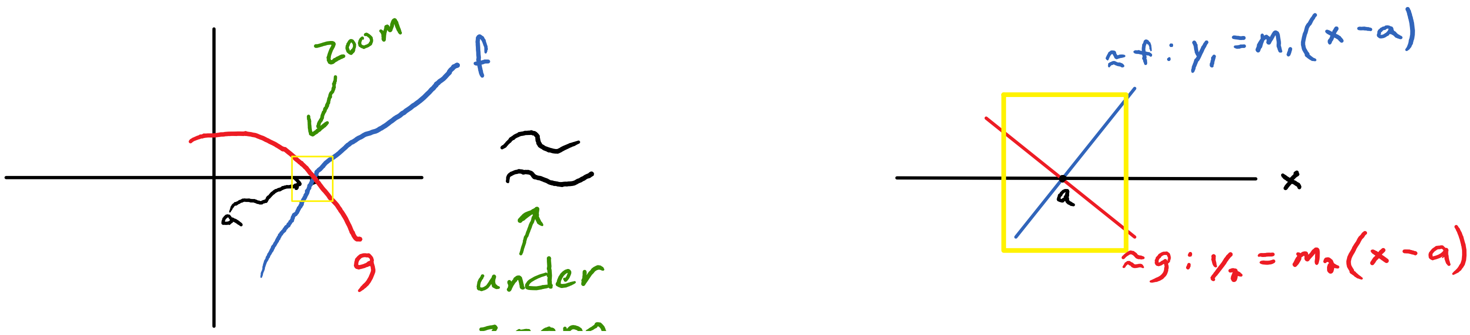
$\frac{0}{0}$ $\{$ $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ (or $\lim\limits_{x\to a} f(x) = \pm\infty$

$\frac{\pm\infty}{\pm\infty}$ $\{$ and $\lim\limits_{x\to a} g(x) = \pm\infty$). Then . . .

indeterminate forms

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{g'(x)}{f'(x)}},$$

Assuming the right hand limit exists or is $\pm\infty$.



g under zoom $\approx g: y_2 = m_2(x-a)$

$$\lim_{x\to a} \frac{f(x)}{g(x)} \approx \lim_{x\to a} \frac{y_1}{y_2} = \lim_{x\to a} \frac{m_1}{m_2} = \lim_{x\to a} \frac{y_1'}{y_2'} \approx \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

ex Evaluate

a) $\displaystyle\lim_{x\to 0} \frac{e^{2x}-1}{x}$ $\quad \frac{0}{0}$ ✓

$$= \lim_{x\to 0} \frac{2e^{2x}}{1} = 2$$

b) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$ $\quad \frac{\infty}{\infty}$ ✓

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Note: Other indeterminate forms include

$$\boxed{0\cdot\infty,\ 1^\infty,\ \infty^0,\ 0^0 \text{ and } \infty-\infty}$$

H doesn't apply directly

ex Evaluate

a) $\displaystyle\lim_{x\to\infty} e^{-x}\sqrt{x}$ $\quad 0\cdot\infty$ H doesn't work

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} \qquad \frac{\infty}{\infty} \ ✓$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x}$$

$$= \lim_{x\to\infty} \frac{1}{2\sqrt{x}\,e^x} = 0$$

b) $$\lim_{x\to 0^+} (\sin x)^x \qquad 0^0 \qquad \text{no } H$$

Set $$y = \lim_{x\to 0^+} (\sin x)^x$$

$$\ln y = \ln\left[\lim_{x\to 0^+} (\sin x)^x\right]$$

$$= \lim_{x\to 0^+} \ln(\sin x)^x$$

fctns are continuous

$$= \lim_{x\to 0^+} x \ln(\sin x) \qquad 0\cdot(-\infty) \qquad \text{No } H$$

limit must be one-sided for this to be true

$$= \lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x} \qquad \frac{-\infty}{\infty} \ ✓$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0^+} \frac{\cot x}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0^+} - \frac{x^2}{\tan x} \qquad \frac{0}{0} \checkmark$$

$$\overset{H}{=} -\lim_{x \to 0^+} \frac{2x}{\sec^2 x} \qquad (H \text{ again!})$$

$$= -\frac{0}{1}$$

$$\ln y = 0$$

$$\log_e y = 0$$

$$y = e^0 = 1$$

$$\text{so} \quad \lim_{x \to 0^+} (\sin x)^x = 1$$

Extra Example (Not done in class)

$$c) \quad \lim_{\left(x \to \frac{\pi}{2}\right)^-} (\sec x - \tan x) \qquad \infty - \infty \quad \text{No } H$$

$$= \lim_{\left(x \to \frac{\pi}{2}\right)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim \left(\frac{1 - \sin x}{} \right) \qquad \frac{0}{0} \checkmark$$

$$= \lim_{\left(x \to \frac{\pi}{2}\right)^-} \left(\frac{1 - \sin x}{\cos x} \right) \qquad \frac{0}{0} \checkmark$$

$$= \lim_{\left(x \to \frac{\pi}{2}\right)^-} \left(\frac{-\cos x}{-\sin x} \right) = \frac{0}{-1} = 0$$