Alternating Series

Goal: To determine if an alternating series converges.

<u>Def:</u> An <u>alternating</u> <u>series</u> has the form $\boxed{\sum_{n=1}^{\infty} (-1)^{n-1} b_n} = b_1 - b_2 + b_3 - b_4 + \cdots$, where $b_n > 0$.

<u>The Alternating Series Test</u> (AST)

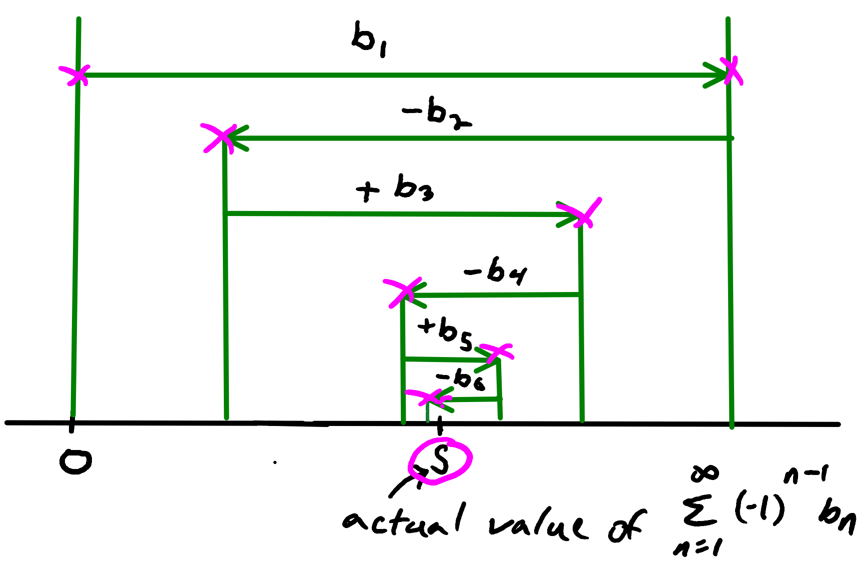
If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, $b_n > 0$, satisfies . . .

$b_{n+1} \le b_n$ $\left\{ \text{① } b_n \ge b_{n+1} \text{ for all } n \right\}$ $b_1 \ge b_2 \ge b_3 \ge \cdots$

② $\lim_{n \to \infty} b_n = 0$

Then the series is convergent.

Pictorially ....



ex Does it converge?

a) $\sum_{n=2} \frac{(-1)^n}{\ln(n)}$

① $b_n = \frac{1}{\ln(n)} > 0$

$b_2 = \frac{1}{\ln(2)} > b_3 = \frac{1}{\ln(3)} > b_4 = \frac{1}{\ln(4)} > \cdots$

So b_n decreases.

② $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$.

So, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges by AST.

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n}{n^2+1}$

$b_n = \frac{n}{n^2+1} > 0$ for $n > 1$.

① Let $f(x) = \frac{x}{x^2+1}$.

$$f'(x) = \frac{\overbrace{(x^2+1)}\cdot 1 - \overbrace{x\cdot 2x}^{2x^2}}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2} < 0, \quad \text{for } x \geq 2$$

f is decreasing for $x \geq 2$.

Thus, b_n is decreasing for $n \geq 2$.

Thus, b_n is decreasing for $n \geq 2$.

abuse of notation

(2) $\lim_{x\to\infty} \frac{x}{x^2+1} \longrightarrow \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{1}{2x} = 0$

For sure dude, then $b_n \to 0$.

Thus the alternating series converges AST.

c) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}}$

$b_n = \frac{\sqrt{n}}{1+2\sqrt{n}} > 0.$

$\lim_{n\to\infty} \frac{\sqrt{n}}{(1+2\sqrt{n})} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\cancel{\frac{1}{\sqrt{n}}}^{\,0} + 2} = \frac{1}{2}$

For large n, $a_n = (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}}$ gets close to two values: $\frac{1}{2}$ or $-\frac{1}{2}$. Thus $a_n \not\to 0$.

So the given series diverges by The Test for Divergence.

Alternating Series Estimation Theorem

If $\boxed{s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n}$ is the sum of a convergent alternating series with $0 \le b_{n+1} \le b_n$, then

$$\underbrace{|R_n|}_{\text{remainder}} = |\underbrace{s - s_n}| \le \underbrace{b_{n+1}}$$

Proof: $s = \underbrace{b_1 - b_2 + b_3 - b_4 + \dots (-1)^{n-1} b_n}_{s_n} + (-1)^n b_{n+1} + (-1)^{n+1} b_{n+2} + \dots$

$$s - s_n = (-1)^n b_{n+1} + (-1)^{n+1} b_{n+2} + (-1)^{n+2} b_{n+3} + \dots$$

$$|s - s_n| = \left|(-1)^n \cdot \left(b_{n+1} - b_{n+2} + b_{n+3} - b_{n+4} + \dots \right)\right|$$

$$|s - s_n| = |b_{n+1} - \underbrace{b_{n+2} + b_{n+3}} - \underbrace{b_{n+4} + {}^{b_{n+5}}} \dots|$$

$$|s - s_n| = |b_{n+1} - \underbrace{(b_{n+2} - b_{n+3})}_{\text{pos.}} - \underbrace{(b_{n+4} - b_{n+5})}_{\text{pos}} + \dots|$$

$$|s - s_n| \le |b_{n+1}|$$

$|s - s_n| \le b_{n+1}$

Done.

(ex) Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$ correct to within two decimal places. (error < $\boxed{0.001}$)

left student: show this converges using AST.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} \approx \overbrace{1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4}}^{\text{Take } s_5 \text{ as estimate}} - \frac{1}{6^4}$$

$b_6 = b_{n+1}$

$\frac{1}{5^4}$: 0.0016 $\frac{1}{6^4}$: $0.00077 < 0.001$.

$$s_5 = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} \approx 0.9475$$

(ex) Approximate to four decimal places

(i.e. $|R_n| = |s - s_n| < 0.00001$): $\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n} \approx \underbrace{-\frac{1}{8} + \frac{2}{8^2} - \frac{3}{8^3} + \frac{4}{8^4} - \frac{5}{8^5} + \frac{6}{8^6}}_{s_n = s_6} - \frac{7}{8^7}$$

$\frac{6}{8^6}$: $0.00002 > .0001$

$\frac{7}{8^7}$: b_{n+1}, $0.000003 < 0.00001$

$s_6 = -0.09876$