Infinite Series

Goal: To determine whether or not an infinite series converges.

Q: How many "parents" do you have going back n generations?

1gen 2gen 3gen 4gen

A: $2^1 + 4 + 8 + 16 + \cdots + 2^n$

$$S_n = 2 + \cancel{4} + \cancel{8} + \cancel{16} + \cancel{\cdots} + \cancel{2^n}$$

$$-2S_n = -(\cancel{4} + \cancel{8} + \cancel{16} + \cancel{\cdots} + \cancel{2^n} + 2^{n+1})$$

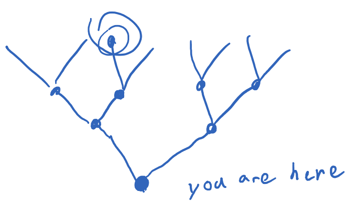
$$S_n - 2S_n = 2 \quad -2^{n+1}$$

$$S_n(1-2) = 2 - 2^{n+1}$$

$$\boxed{S_n = 2^{n+1} - 2} \ *$$

Let $n = 10$: $S_{10} = 2046$ parents

$n = 20$: $S_{20} = 2{,}097{,}150$ parents



$\sum_{i=1}^{n} 2^i$ is called a <u>finite geometric series</u> and

$a_i = 2^i$ is a <u>geometric sequence</u>

with <u>common</u> <u>ratio</u> <u>$r = 2$</u>.

<u>Definitions</u>

① $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ is called an <u>infinite series</u>

② The <u>nth partial sum</u> of an infinite sequence $\{a_k\}_{k=1}^{\infty}$ is given by

$S_n =$ <u>$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$</u>

③ The sequence of partials sums is denoted by $\{S_n\}_{n=1}^{\infty}$

④ If S_n converges to a real number S, we say that the series $\boxed{\sum_{n=1}^{\infty} a_n}$ <u>converges</u> to S (so, $a_1 + a_2 + a_3 + \cdots = S$).

[i.e. if $\sum_{n=1}^{\infty} a_n$ converges, then

$$\sum_{n=1}^{\infty} a_n = \boxed{\lim_{n\to\infty} S_n} = S \;]$$

$$\sum_{n=1}^{\infty} a_n = \boxed{\lim_{n\to\infty} S_n} = S]$$

ex Find the first 3 partial sums of $\boxed{a_n = \frac{1}{2^{n-1}}}$, $n = 1, 2, 3, \ldots$

$a_1 = \frac{1}{2^0} = 1$

$a_2 = \frac{1}{2}$

$a_3 = \frac{1}{4}$

$S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1 + \frac{1}{2}$

$S_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4}$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = S = a_1 + a_2 + a_3 + \cdots$$

ex Decide if $\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)}_{\text{telescoping series}}$ converges. If so, find the sum.

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) =$$

$$= \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{5}} - \tfrac{1}{7}\right) + \left(\tfrac{1}{7} - \tfrac{1}{9}\right) + \cdots$$

$$S_n = \sum_{i=1}^{n}\left(\frac{1}{2i-1} - \frac{1}{2i+1}\right)$$

$$S_n = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{7}}\right) + \left(\cancel{\tfrac{1}{7}} - \cancel{\tfrac{1}{9}}\right) + \cdots + \left(\cancel{\tfrac{1}{2n-1}} - \tfrac{1}{2n+1}\right)$$

$$S_n = 1 - \frac{1}{2n+1}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \lim_{n\to\infty}\left(1 - \cancel{\frac{1}{2n+1}}^{\,0}\right)$$

$$= 1$$

ex Decide if $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$ converges

$$a_n = \frac{2}{4n^2-1}$$

$\boxed{\text{PFD}}$

$$\left[\frac{2}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}\right](2n-1)(2n+1)$$

$$2 = A(2n+1) + B(2n-1)$$

$\boxed{n = \frac{1}{2}}$

$2 = A(2)$

$A = 1$

$\boxed{n = -\frac{1}{2}}$

$2 = B(-2)$

$B = -1$

$$\frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}$$

$2 = A(2)$

$A = 1$

$2 = B(-2)$

$B = -1$

$$\sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \sum_{n=1}^{\infty} \left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$\vdots$ see last ex

$= 1$

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $\underbrace{a_n \to 0}_{\lim_{n\to\infty} a_n = 0}$.

works since if $a_n \not\to 0$, then the series adds an infinite # of non-diminishing terms, which can never converge to a single #.

nth Term Test for Divergence

If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

Warning: only use to show divergence, not convergence.

ex Decide if $\sum_{n=1}^{\infty} \frac{n}{2n+3}$ converges

$a_n = \frac{n}{2n+3} \to \frac{1}{2} \neq 0$. So, the series diverges by the nth term test.

Def: The series given by $\sum_{n=1}^{\infty} ar^{n-1}$ is called a geometric series with ratio r.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

For a finite geometric series,

$$S_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}$$

use a technique like the one that generate $*$ to get the sum for S_n.

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

$\to 0, \; -1<r<1$

Sum for S_n.

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

$$S = \sum_{n=1}^{\infty} a r^{n-1} = \lim_{n\to\infty} \frac{a(1-\overset{0,\ -1<r<1}{\cancel{r^n}})}{1-r}, \text{ if it exists.}$$

$$= \frac{a}{1-r}$$

Theorem: An infinite geometric series converges to $\frac{a}{1-r}$ as long as $|r|<1$ and diverges otherwise.

(i.e. $\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$ for $-1<r<1$)

ex Does it converge? If so, find the sum

a) $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$

$-1 < r = \frac{1}{3} < 1, \quad a = \left(\frac{1}{3}\right)^0 = 1$

$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}}$

$= \frac{1}{\frac{2}{3}}$

$= \frac{3}{2}$

b) $\sum_{n=0}^{\infty} 2\left(-\frac{3}{4}\right)^n$

$-1 < r = -\frac{3}{4} < 1$

$a = 2$

$S = \frac{2}{1-\left(-\frac{3}{4}\right)}$

$= \frac{2}{\frac{7}{4}}$

$= \frac{8}{7}$

c) $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots$

$-1 < r = \frac{-1}{2} < 1, \quad a = 2$

$S = \frac{a}{1-r}$

$= \frac{2}{1+\frac{1}{2}}$

$= \frac{2}{\frac{3}{2}} = \frac{4}{3}$

d) $\sum_{n=0}^{\infty} 2(-1.03)^n$

$r = -1.03 < -1$

The series doesn't converge

ex $0.\overline{9} = 1$

$0.\overline{9} = 0.999999999$

$= 0.9 + 0.09 + 0.009 + \cdots$

$r = \frac{1}{10} = .1 \quad a = 0.9$

$$S = \frac{a}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$$

Notes: Assuming both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge . . .

① $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

② $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

Careful when using ②! Both individual series must converge for ② to be true

ex

$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + \cdots$

$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 + \cdots$

both series diverge

but . . .

$\sum_{n=1}^{\infty} \left[\overbrace{(-1)^{n+1} + (-1)^n}^{0} \right] = 0$ converges