Calculus with Parametric Curves

Let $(f(t), g(t))$ define a parametric curve with $f'(t)$ and $g'(t)$ continuous.

Eliminating the parameter gives $y = F(x)$ or

$g(t) = F[f(t)]$.

$\underbrace{y'}_{\text{w.r.t. } t} = g'(t) = F'[f(t)] \cdot f'(t)$

$$F'[f(t)] = \frac{g'(t)}{f'(t)}$$

$$F'(x) = \frac{g'(t)}{f'(t)}$$

★ $$\boxed{\frac{dy}{dx} = F'(x) = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

ex Let $x = 2t^2+1$, $y = \frac{1}{3}t^3 - t$

a) Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2-1}{4t}$$

b) Find the eqn. of the tangent line at $t=3$.

$$m = \left.\frac{dy}{dx}\right|_{t=3} = \frac{3^2-1}{4(3)} = \frac{8}{12} = \frac{2}{3}$$

$$\boxed{m = \frac{2}{3}}$$

Point: $x = 2(3^2)+1 = 19$, $y = \frac{1}{3}(3)^3 - 3 = 6$ $\Big\}$ $(19,6)$

$$y - y_1 = \underbrace{m}_{\left.\frac{dy}{dx}\right|_{t=3}}(x - x_1)$$

$$y - 6 = \frac{2}{3}(x - 19) \rightarrow y = \frac{2}{3}(x-19)+6$$

let $x_2 = t$

$$y_2 = \frac{2}{3}(t-19)+6$$

c) Find $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \boxed{\dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}}$$

☆ formula for $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx} = \frac{t^2-1}{4t}$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{t^2-1}{4t}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{4t(2t)-(t^2-1)4}{16t^2}}{4t}$$

$$= \frac{\cancel{4}\left(2t^2-t^2+1\right)}{\cancel{4}\cdot 16t^3}$$

$$= \frac{t^2+1}{16t^3}$$

(ex) Find the points where the tangent line is horizontal or vertical:

$$x = 2t^3 + 3t^2 - 12t, \quad y = 2t^3 + 3t^2 + 1$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

H-tan: set $\frac{dy}{dt} = 0$ V-tan: set $\frac{dx}{dt} = 0$

$$\frac{dy}{dt} = \boxed{6t^2 + 6t = 0}$$

$$6t(t+1) = 0$$

$$t = 0 \text{ or } t = -1$$

plug in to find (x,y)

$$x = 2t^3 + 3t^2 - 12t, \quad y = 2t^3 + 3t^2 + 1$$

$t = 0$

$x = 0$

$y = 1$

$t = 1$

$x = -2 + 3 + 12 = 13$

$y = -2 + 3 + 1 = 2$

$(0,1)$ $(13,2)$

$$\frac{dx}{dt} = 6t^2 + 6t - 12 = 0$$

$$t^2 + 1t - 2 = 0$$

$$(t-1)(t+2) = 0$$

$$t = 1, \quad t = -2$$

$x = 2 + 3 - 12$

$x = -7$

$y = 2 + 3 + 1$

$y = 6$

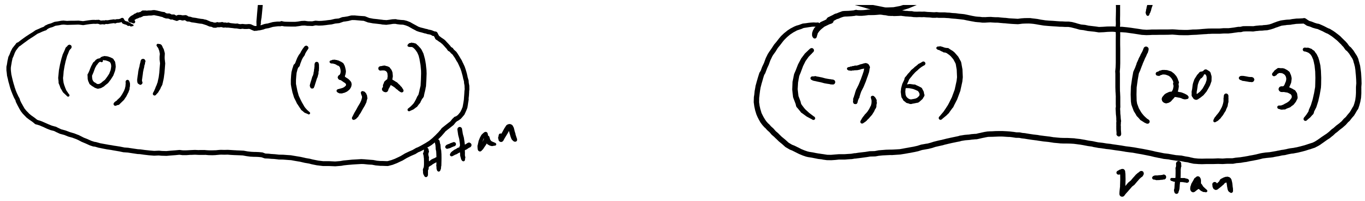
$x = -16 + 12 + 24$

$x = 20$

$y = -16 + 12 + 1$

$y = -3$

$(-7, 6)$ $(20, -3)$



Note: If $\dfrac{dy/dt}{dx/dt} = \dfrac{0}{0}$ at $t=a$, then it may not be clear whether or not graph has a vertical or horizontal tangent at $(f(a), g(a))$.

You may have to take $\lim\limits_{t\to a} \dfrac{dy/dt}{dx/dt}$ to see if $(f(a), g(a))$ is a H-tan or V-tan

Area

Let $y = F(x)$, $F(x) \geq 0$ and $x = f(t)$, $y = g(t)$.

If the curve given by F is traced out once as t goes from α to β, then

$$\boxed{\text{Area} = \int_a^b F(x)\,dx = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt}$$

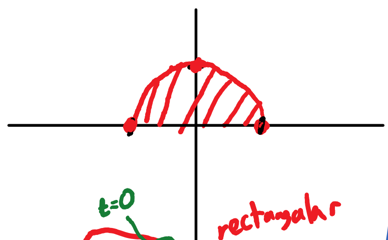
where $a = f(\alpha)$, $b = f(\beta)$

ex Find the area between the curve and the x-axis.

curve: $x = \cos t$, $y = \sin^2 t$

$$\text{Area} = \int_{\pi}^{0} \sin^2 t\,(-\sin t)\,dt$$

t	x	y
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0



rectangular

$$A = \int_{-1}^{1} f(x)\,dx$$

t=0

t=π

Time-out

$$\int \sin^2 t\,(-\sin t)\,dt$$

$u = \cos t$

$$\int (1-\cos^2 t)\underbrace{(-\sin t)\,dt}_{du}$$

$$\cos t - \frac{\cos^3 t}{3} + C$$

Time-In

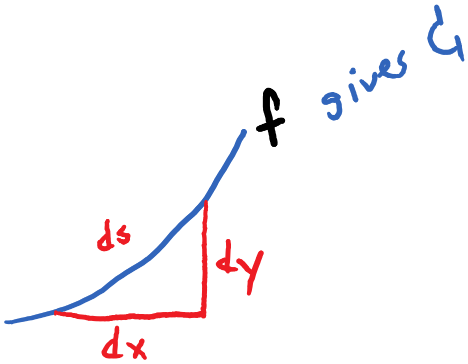
$$\left[\cos t - \frac{\cos^3 t}{3}\right]_{\pi}^{0}$$

$$\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$$

$$= 2 - \frac{2}{3}$$

$$= \frac{4}{3}$$

Arc Length



$$ds = \sqrt{dx^2 + dy^2}\ \frac{dt}{dt}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$\boxed{S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt}$$

Notes: (1) $f'(t)$, $g'(t)$ continuous
(2) C traced out once
(3) C doesn't intersect itself
(except maybe at endpoints)

(ex) Find the arc length of $x = t^2$, $y = 2t$ $\quad 0 \le t \le 2$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2$$

$$S = \int_0^2 \sqrt{4t^2 + 4}\ dt$$

$$= 2\int_0^2 \sqrt{t^2+1}\ dt$$

$$\overset{\#21}{=} 2\left[\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\ln\left(t + \sqrt{t^2+1}\right)\right]_0^2$$

$$= 2\left[\sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)\right]$$

8.1

ex) Find the arclength of $\boxed{y = x^{\frac{3}{2}} - 1}$ on $\underbrace{[0,4]}_{0 \le x \le 4}$.

$$\boxed{\text{let} \quad x = t \qquad y = t^{\frac{3}{2}} - 1, \qquad 0 \le t \le 4}$$

parameterize the curve

$y = F(x)$

let $x = t$, $y = F(t)$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \frac{3}{2} t^{\frac{1}{2}}$$

$$ds = \sqrt{1^2 + \left(\frac{3}{2} t^{\frac{1}{2}}\right)^2} = \sqrt{1 + \frac{9}{4} t}\; dt$$

$$s = \int_0^4 \sqrt{1 + \frac{9}{4} t}\; dt$$

$u = 1 + \frac{9}{4} t$

$\vdots$

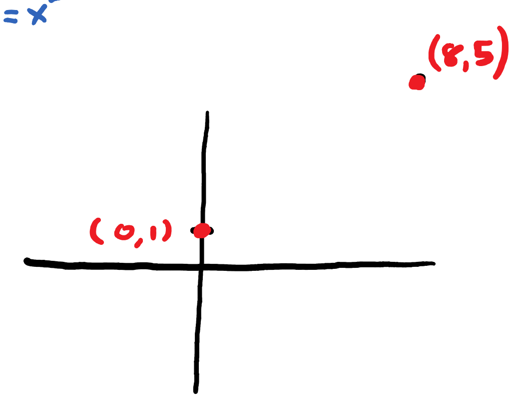
$$\frac{8}{27}\left(10\sqrt{10} - 1\right)$$

ex) Find arclength of $(y-1)^3 = x^2$ from $(0,1)$ to $(8,5)$

$$x^2 = (y-1)^3$$

$$x = \pm (y - 1)^{\frac{3}{2}}$$

$$x = \pm (5 - 1)^{\frac{3}{2}}$$



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use $\boxed{x = (y-1)^{\frac{3}{2}}}$ $\rightarrow$ let $y = t$

$x = (t-1)^{\frac{3}{2}}$

$x = (t-1)^{\frac{3}{2}} \qquad y = t \quad , \quad 1 \le t \le 5$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = \frac{3}{2}(t-1)^{\frac{1}{2}}, \qquad \frac{dy}{dt} = 1$$

$$ds = \sqrt{\frac{9}{4}(t-1) + 1}\, dt$$

$$= \sqrt{\frac{9}{4}t - \frac{9}{4} + 1}\, dt$$

$$ds = \sqrt{\frac{9}{4}t - \frac{5}{4}}\, dt$$

$$s = \int_1^5 \sqrt{\frac{9}{4}t - \frac{5}{4}}\, dt$$

$$= \frac{8}{27}\left[10\sqrt{10} - 1\right]$$