Complex Numbers and WAVES

I. Complex Numbers

Informal Defn:

Let $i = $ "$\sqrt{-1}$" the "imaginary unit"
Note $i^2 = -1$

The complex numbers, denoted by $\mathbb{C}$

is the set of all $a+bi$
where a and b are in $\mathbb{R}$,
with operations

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

$$(a+bi)\cdot(c+di)=(ac-bd)+(ad+bc)i$$

But does i exist?

Formal Defn:

The complex numbers ($\mathbb{C}$)
is the set of all ordered pairs (a,b)
where a,b are in $\mathbb{R}$,
with operations

$$(a,b)+(c,d)=(a+c,\ b+d)$$

$$(a,b)\cdot(c,d)=(ac-bd,\ ad+bc)$$

That's all! We don't have to assume i exists!

From the formal definition, can prove

Theorem: $\mathbb{C}$ is a field.

Specifically, $+$ and $\cdot$ are commutative, associative,
$\cdot$ distributes over $+$,
$(0,0)$ is the additive identity,
$(1,0)$ is the multiplicative identity,
every (a,b) has an additive inverse:

$$-(a,b) = (-a, -b)$$

every $(a,b)\neq 0$ has a mult. inverse:

$$(a,b)^{-1}=\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

Excerpt from proof:

- distributive over $+$.

$$\underbrace{(a,b)\cdot((c,d)+(e,f))}=\underbrace{(a,b)\cdot(c,d)+(a,b)\cdot(e,f)}$$

$$=(a,b)(c+e,\ d+f)$$

$$=(a(c+e)-b(d+f),\ a(d+f)+b(c+e))$$

$$=(ac+ae-bd-bf,\ ad+af+bc+be)$$

$$(ac-bd,\ ad+bc)+(ae-bf,\ af+be)$$

$$=(ac-bd+ae-bf,\ ad+bc+af+be)$$

these ordered pairs are equal!

Shorthand notation:

$$(a,0)\to a$$

$$(0,1)\to i$$

thus $\underbrace{(a,b)\to a+bi}$

why? $(a,b)=(a,0)+(0,b)$
$=(a,0)+(b,0)\cdot(0,1)$
$\to a + b\cdot i$

Note

$$i^2 = (0,1)\cdot(0,1)$$

$$=(0\cdot 0-1\cdot 1,\ 0\cdot 1+1\cdot 0)$$

$$=(-1,0)\to -1$$

that is, $\boxed{i^2=-1}$! Thus $i=\sqrt{-1}$

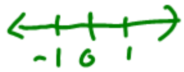
Notation: We often denote a complex number by a single letter, such as z, w, v

Defn: If $z = a + bi$, we write $a = \operatorname{Re} z$ (the "real part" of z), $b = \operatorname{Im} z$ (the "imaginary part" of z)

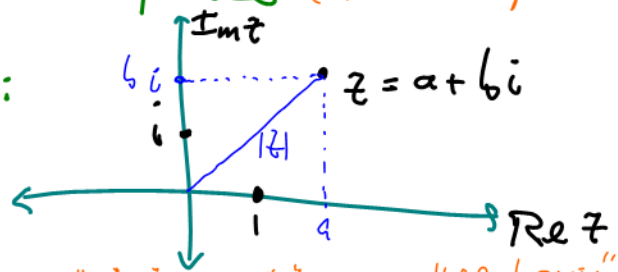
$$\underbrace{|z|}_{\text{the "magnitude" or "modulus" of } z} = \sqrt{a^2 + b^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$$

Geometric models for $\mathbb{R}$ and $\mathbb{C}$.

$\mathbb{R}$: line (the "number line")



$\mathbb{C}$: plane (the "complex plane")

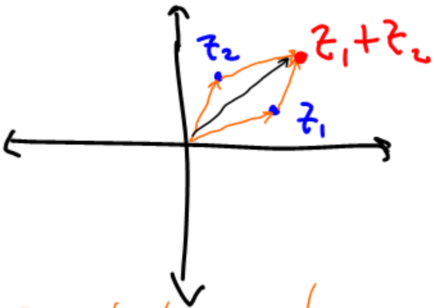


"imaginary axis": all $(0,b)$

"real axis": all $(a,0)$

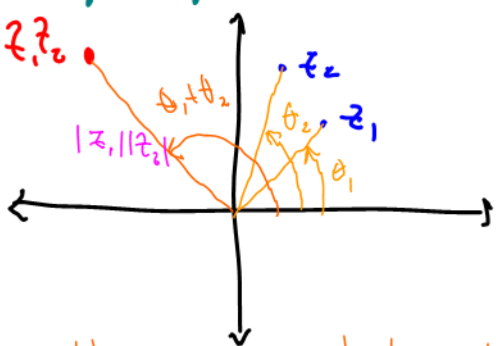
Addition and multiplication in the complex plane

Addition



Parallelogram law (same for vectors in $\mathbb{R}^2$)

Multiplication

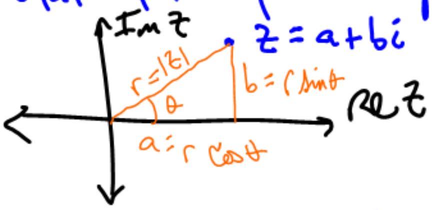


multiply magnitudes: $|z_1 z_2| = |z_1||z_2|$

add angles: angle of $z_1 z_2$ = angle of z_1 + angle of z_2

Why does mult. work this way?

Polar form of a complex number:



$$z = \underbrace{r\cos\theta}_{a} + i\,\underbrace{r\sin\theta}_{b}$$

$$= r(\cos\theta + i\sin\theta)$$

Rect → Polar: $r = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$

$\tan\theta = \cancel{\tan^{-1}} \left(\operatorname{Im} z / \operatorname{Re} z\right)$

Polar → Rect: $\operatorname{Re} z = r\cos\theta$

$\operatorname{Im} z = r\sin\theta$

check

$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 \big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$+ i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big)$

$= r_1 r_2 \big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)$

= complex # w/ amplitude $r_1 r_2$ and angle $\theta_1 + \theta_2$ ✓

Wait!

$$\begin{pmatrix}\text{complex \# at}\\ \text{angle } \theta_1\end{pmatrix}\cdot\begin{pmatrix}\text{complex \# at}\\ \text{angle } \theta_1\end{pmatrix} = \begin{pmatrix}\text{complex \# at}\\ \text{angle } \theta_1+\theta_2\end{pmatrix}$$

multiplication turns into addition
Thus θ is behaving like an exponent!

More precisely: $\cos\theta + i\sin\theta$ behaves like an exponential function!
– call it $B^\theta = \cos\theta + i\sin\theta$

why?

$$\underbrace{(\cos\theta_1 + i\sin\theta_1)}_{B^{\theta_1}}\cdot\underbrace{(\cos\theta_2 + i\sin\theta_2)}_{B^{\theta_2}} = \underbrace{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)}_{B^{\theta_1+\theta_2}}$$

$$\underbrace{B^{\theta_1}\cdot B^{\theta_2} = B^{\theta_1+\theta_2}}_{\text{product rule for exponents}}$$

What is B?

set $\theta = 1$: $B = B^1 = \cos 1 + i\sin 1$

Wait! Isn't power series of $\sin x$, $\cos x$ related to e^x?

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \quad \text{(odd powers)}$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \quad \text{(even powers)}$$

$$e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n \quad \text{(all powers)}$$

Let's try it!

$$\cos 1 + i\sin 1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(1)^{2n} + i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(1)^{2n+1}$$

Note $-1 = i^2$
so $(-1)^n = (i^2)^n = i^{2n}$
$i(-1)^n = i^{2n+1}$

$$= \underbrace{\sum_{n=0}^{\infty}\frac{i^{2n}}{(2n)!}}_{\text{even}} + \underbrace{\sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)!}}_{\text{odd}} = \sum_{n=0}^{\infty}\frac{i^n}{n!} = e^i$$

Thus $\boxed{\cos\theta + i\sin\theta} = B^\theta = (e^i)^\theta = \boxed{e^{i\theta}}$ Euler's identity

Note: $e^{i\pi} = \cos\pi + i\sin\pi = -1$!

Note – a more formal logically sound approach is to prove this directly exactly as we did above, replacing 1 by θ.

Note: $\mathrm{Re}\, e^{i\theta} = \cos\theta$, $\mathrm{Im}\, e^{i\theta} = \sin\theta$

Summary: A complex number z can be written in..

Standard (Rectangular) Form

$$\boxed{z = a + bi} \quad (a = \mathrm{Re}\, z,\ b = \mathrm{Im}\, z)$$

Polar Form

$$\boxed{z = re^{i\theta}}$$

$r = |z| = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\frac{b}{a}$

$a = r\cos\theta$, $b = r\sin\theta$

II. Waves

Mathematical representation of pure sinusoidal wave traveling $+x$ direction with a fixed frequency and wavelength λ:

$$y = \varphi(x,t) = a\cos(\omega t - kx + \delta)$$

height of wave at position x and time t

a: amplitude (max height)

$\omega = \frac{2\pi}{\nu}$, $k = \frac{2\pi}{\lambda}$

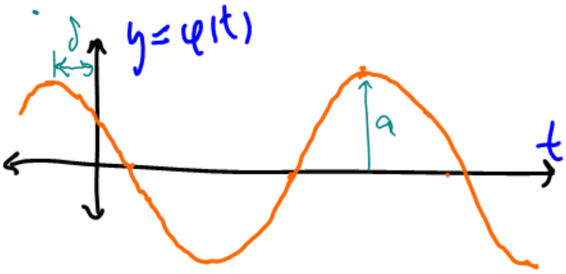
δ: phase shift

To simplify our analysis, we will fix $x = 0$ (i.e. we will track the height of wave at a single point in space) and let $\omega = 1$

Thus at $x = 0$:

$$\boxed{y = \varphi(t) = a\cos(t+\delta)}$$

real-valued wave



Note: $\varphi(t) = a\cos(t+\delta) = \mathrm{Re}\left[a e^{i(t+\delta)}\right] = \mathrm{Re}\left[\underbrace{ae^{i\delta}}_{A} e^{it}\right] = \mathrm{Re}\,\Phi(t)$

where $\boxed{\Phi(t) = Ae^{it}}$

complex-valued wave

$ae^{i\delta}$ – "complex amplitude"

Summary: Two representations of a wave

Real-valued:

$$\varphi(t) = a\cos(t+\delta)$$

Complex-Valued:

$$\underline{\Phi}(t) = A e^{it} \qquad \underbrace{A = a e^{i\delta}}_{\text{"Complex amplitude"}}$$

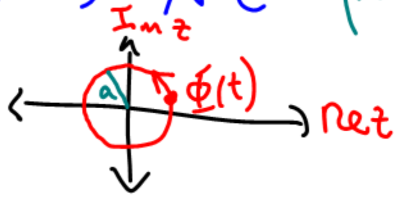
"Complex amplitude" encodes both amplitude AND phase shift

$$\varphi(t) = \mathrm{Re}\,\underline{\Phi}(t)$$

Graphical Representations

Complex-Valued Wave

$$\underline{\Phi}(t) = A e^{it} \quad (A = a e^{i\delta})$$



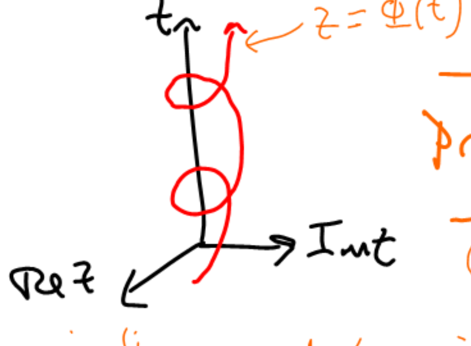
$\Phi(t)$ rotates ccw around circle of radius a in complex plane.

Real-Valued Wave

$$\varphi(t) = a\cos(t+\delta)$$

$\varphi(t)$ moves back and forth between $-a$ and a on real line.

3D picture:

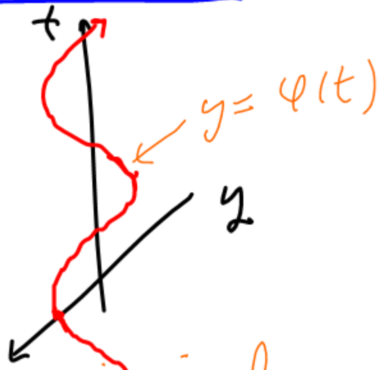


spiral around t-axis

Project by taking Real Part

(project onto Re z, t plane)

2D picture



sinusoid

III. Superposition of Waves

We wish to add two waves:

$$\varphi(t) = \underbrace{\varphi_1(t)}_{a_1\cos(t+\delta_1)} + \underbrace{\varphi_2(t)}_{a_2\cos(t+\delta_2)} = \underbrace{a}_{a=?}\cos(t+\underbrace{\delta}_{\delta=?})$$

$a = a_1 + a_2$?

We will add φ_1 and φ_2 in two ways:

Real-Valued Waves

$$\varphi_1(t) = a_1 \cos(t+\delta_1)$$
$$\varphi_2(t) = a_2 \cos(t+\delta_2)$$
$$\varphi(t) = \varphi_1(t) + \varphi_2(t)$$

want

$$= a\cos(t+\delta)$$
$$= a(\cos t\cos\delta - \sin t\sin\delta)$$
$$= (a\cos\delta)\cos t - (a\sin\delta)\sin t$$

do the algebra!

$$= a_1(\cos t\cos\delta_1 - \sin t\sin\delta_1) + a_2(\cos t\cos\delta_2 - \sin t\sin\delta_2)$$
$$= (a_1\cos\delta_1 + a_2\cos\delta_2)\cos t - (a_1\sin\delta_1 + a_2\sin\delta_2)\sin t$$

equate coefficients of $\cos t$ and $\sin t$

so
$$a\cos\delta = a_1\cos\delta_1 + a_2\cos\delta_2$$
$$a\sin\delta = a_1\sin\delta_1 + a_2\sin\delta_2$$

square, add, and take $\sqrt{\ }$:

$$\boxed{a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\delta_1-\delta_2)}}$$

divide, take $\tan^{-1}$:

$$\boxed{\delta = \tan^{-1}\left(\frac{a_1\sin\delta_1 + a_2\sin\delta_2}{a_1\cos\delta_1 + a_2\cos\delta_2}\right)}$$

Messy!
a, δ are complicated nonlinear functions of a_1, a_2, δ_1 and δ_2

$a \neq a_1 + a_2$ (unless $\delta_1 = \delta_2$)

Real amplitudes do not add!

Note:
$$\varphi_1(t) = \mathrm{Re}\,\Phi_1(t)$$
$$\varphi_2(t) = \mathrm{Re}\,\Phi_2(t)$$
$$\varphi(t) = \mathrm{Re}\,\Phi(t)$$

Complex Valued Waves

$$\Phi_1(t) = A_1 e^{it} \qquad A_1 = a_1 e^{i\delta_1}$$
$$\Phi_2(t) = A_2 e^{it} \qquad A_2 = a_2 e^{i\delta_2}$$
$$\Phi(t) = \Phi_1(t) + \Phi_2(t)$$

want

$$A e^{it}$$

do the algebra

$$A_1 e^{it} + A_2 e^{it}$$
$$= (A_1 + A_2)e^{it}$$

equate coefficients of e^{it}

$$\boxed{A = \underbrace{A_1 + A_2}}$$

amplitude of superposition is sum of the individual amplitudes

Simple!
A is a simple linear combination of $A_1 + A_2$

Complex amplitudes add!

Wait! Since $\varphi(t) = \mathrm{Re}\,\Phi(t)$ then we should be able to recover the above formulas for a and δ in terms of a_1, a_2, δ_1, and δ_2 from the fact that $A = A_1 + A_2$!

Let's do it!

$$\Phi(t) = (A_1 + A_2)\, e^{it} \qquad A_1 = a_1 e^{i\delta_1}, \; A_2 = a_2 e^{i\delta_2}$$

$$\begin{aligned}
A_1 + A_2 &= a_1 e^{i\delta_1} + a_2 e^{i\delta_2} \\
&= a_1(\cos\delta_1 + i\sin\delta_1) + a_2(\cos\delta_2 + i\sin\delta_2) \\
&= (a_1\cos\delta_1 + a_2\cos\delta_2) + i(a_1\sin\delta_1 + a_2\sin\delta_2)
\end{aligned}$$

$$\begin{aligned}
|A_1 + A_2| &= \sqrt{(\qquad)^2 + (\qquad)^2} \\
&= \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\delta_1 - \delta_2)} = a
\end{aligned}$$

(from above)

$$\text{angle of } A_1 + A_2 = \tan^{-1}\frac{\operatorname{Im}(A_1+A_2)}{\operatorname{Re}(A_1+A_2)} = \tan^{-1}\left(\frac{a_1\sin\delta_1 + a_2\sin\delta_2}{a_1\cos\delta_1 + a_2\cos\delta_2}\right)$$

$$= \delta \text{ (from above)}$$

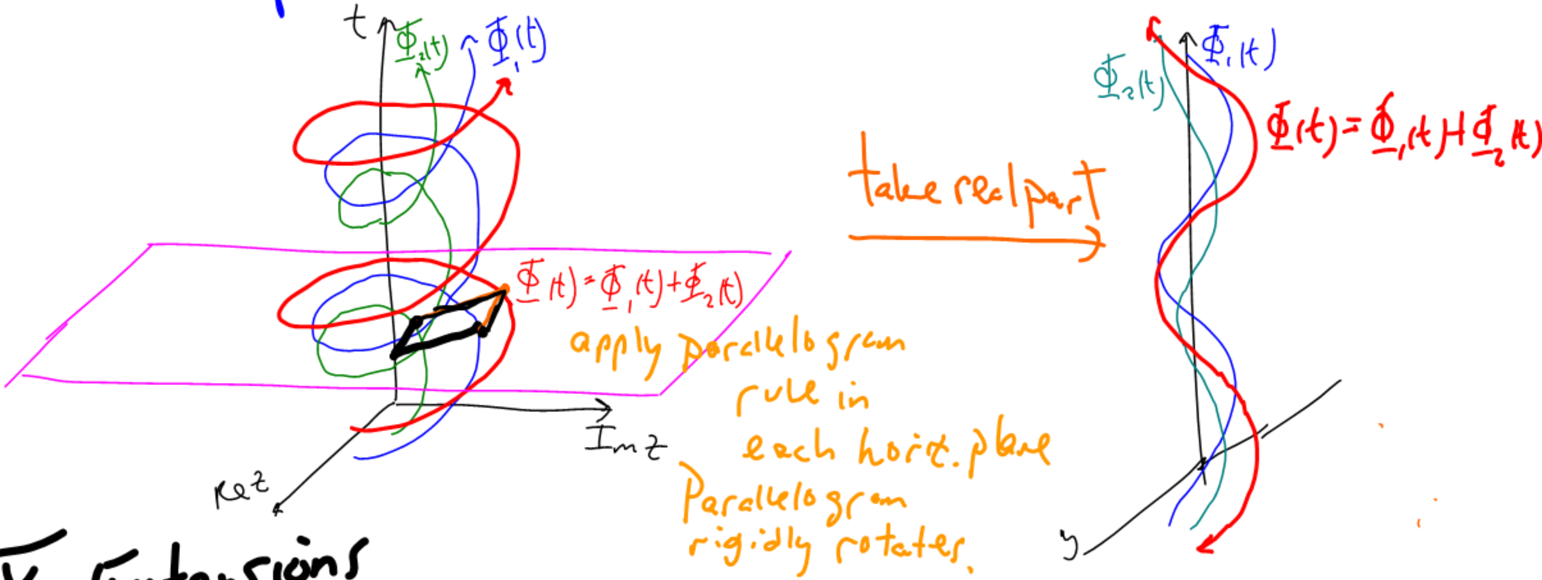
$\therefore$ Polar form of $A_1 + A_2 = a e^{i\delta}$

$$\text{thus } \Phi(t) = a e^{i\delta} e^{it} = a e^{i(t+\delta)}$$

Note $\operatorname{Re}\Phi(t) = a\cos(t+\delta) = \varphi(t)$ ✓

Graphical representation of superposition of two pure sinusoids

Complex-valued waves — Real Valued Waves



IV Extensions

1) spatial ($\vec{r}=(x,y,t)$) dependence, general wavelength and frequency

real-valued wave $\quad \varphi(t) = a\cos(\vec{k}\cdot\vec{r}-\omega t+\delta) = \mathrm{Re}\,\underbrace{\Phi(t)}_{a e^{i\delta}e^{i(\vec{k}\cdot\vec{r}-\omega t)}}$ complex-valued wave

wave travels in direction $\vec{k}$ with wavelength λ, $|\vec{k}|=\frac{2\pi}{\lambda}$

$\omega = \frac{2\pi}{\nu}$, ν = freq.

2) Another approach to $\mathbb{C}$:

$\mathbb{C}$ = all 2x2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

write $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \to a \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to i$ (Note: $i^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \to -1$)

$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \to a+bi$

with usual matrix ops, can easily show $\mathbb{C}$ is closed under + and ×, and all field axioms hold!

3) Waves that are combinations of different frequencies and wavelengths

$$\sum_j \underbrace{A_j}_{\text{complex amplitudes}} e^{i(k_j \cdot \vec{r} - \omega_j t)} \quad \text{or} \quad \int_{-\infty}^{\infty} dk \, \underbrace{A(k)}_{\text{complex amplitude function}} e^{i(k \cdot \vec{r} - \omega_k t)}$$

Discrete Fourier Representation of Wave

Continuous Fourier representation of Wave