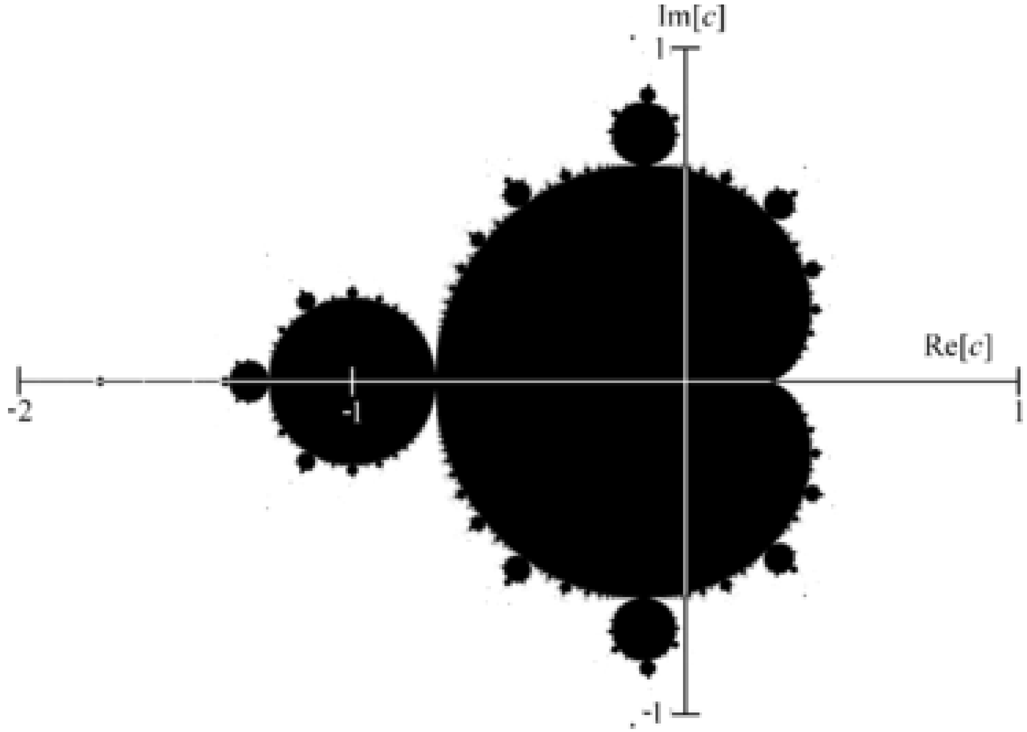


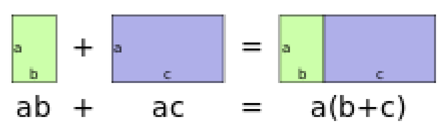
Complex numbers and Trigonometric Identities

The shortest path between two truths in the real domain passes through the complex domain.

Jacques Hadamard

Simplicity in linearity

- In Mathematics, we know that the distributive property states:
- a(b + c) = ab + ac
- But why is this even true to begin with?
- Here is a visual proof where we can think of the real number values representing the lengths of rectangles and their products the area of their associated rectangles.



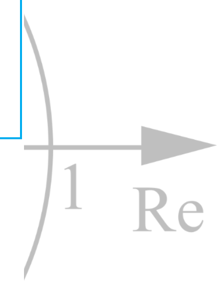
- Even the proof for natural numbers takes effort.
- Since $m \cdot n$ is just $n + n + \cdots + n$, repeated m times. Then by using the commutative property: $a(b + c) = (b + c)a = \underbrace{a + a + \cdots + a}_{b+c\ times}$

$$\underbrace{(a + a + \cdots + a)}_{b\ times} + \underbrace{(a + a + \cdots + a)}_{c\ times} = b \cdot a + c \cdot a = ab + bc$$

https://math.stackexchange.com/questions/786066/prove-the-distributive-law-abc-abac-for-real-numbers

Simplicity in linearity

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

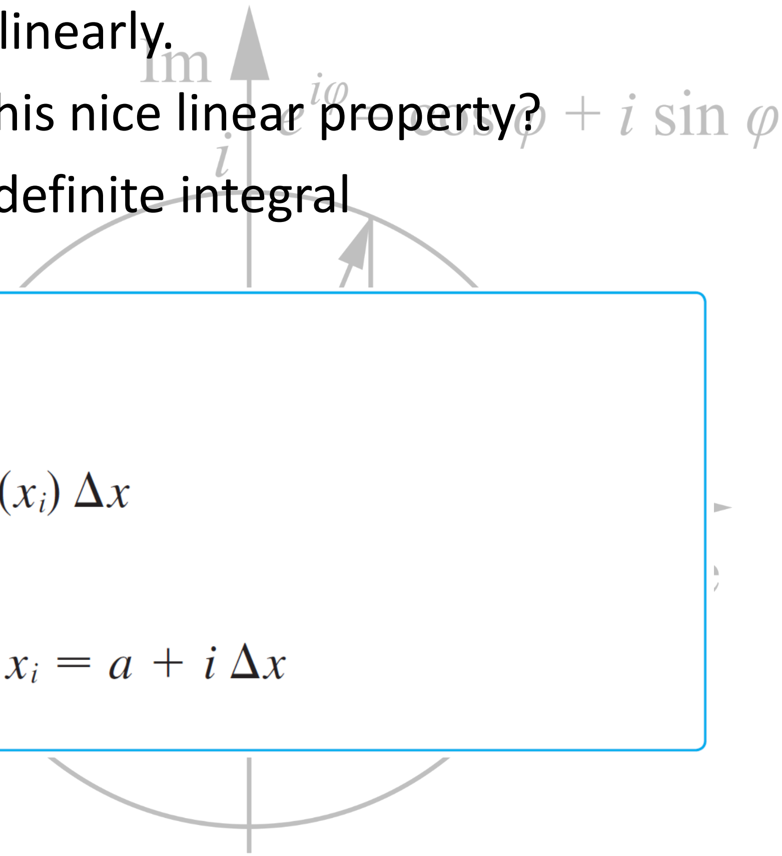
$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Proof We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\begin{aligned}
\frac{d}{dx}[u(x) + v(x)] &= \lim_{h \to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\
&= \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right] \\
&= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}. \quad \blacksquare
\end{aligned}$$

Its definitively ALIVE!!!

- Previous theorem show how you will see in Calculus 1 how the derivative of two functions does behave linearly.
- What other mathematical objects have this nice linear property?
- Lets take another result from calc 1, the definite integral



4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

Simplicity in Linearity

- Linearity property of the definite integral

$$\int_a^b [f(x) + g(x)]\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\,\Delta x$$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} f(x_i)\,\Delta x + \sum_{i=1}^{n} g(x_i)\,\Delta x \right]$$

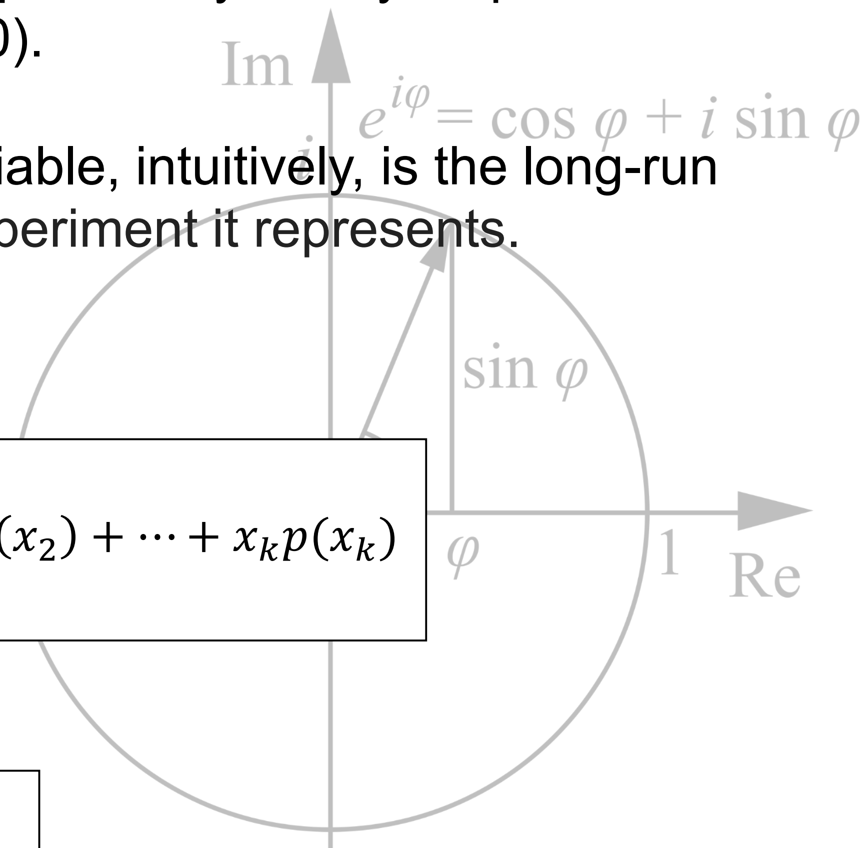
$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x + \lim_{n \to \infty} \sum_{i=1}^{n} g(x_i)\,\Delta x$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

As expected…

Lets go beyond calculus, and go into probability theory. Expect to see this in a statistics class (Math 120).

The **expected value** of a random variable, intuitively, is the long-run average value of repetitions of the experiment it represents.



Discrete case:

$$E[X] = \sum_{i=1}^{k} x_i p(x_i) = x_1 p(x_1) + x_2 p(x_2) + \cdots + x_k p(x_k)$$

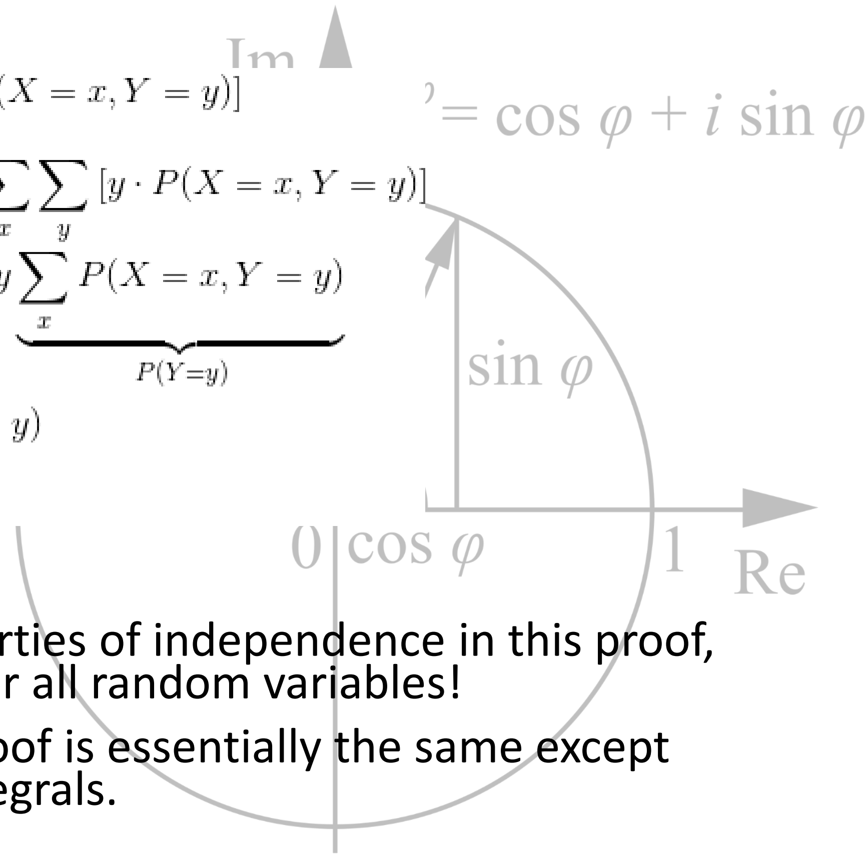
Continuous case:

$$E[X] = \int_{\mathbb{R}} x p(x)\, dx$$

As expected... a proof

- For discrete random variables X and Y. By the basic definition of expected value,

$$
\begin{aligned}
E[X+Y] &= \sum_x \sum_y [(x+y)\cdot P(X=x, Y=y)] \\
&= \sum_x \sum_y [x \cdot P(X=x, Y=y)] + \sum_x \sum_y [y \cdot P(X=x, Y=y)] \\
&= \sum_x x \underbrace{\sum_y P(X=x, Y=y)}_{P(X=x)} + \sum_y y \underbrace{\sum_x P(X=x, Y=y)}_{P(Y=y)} \\
&= \sum_x x \cdot P(X=x) + \sum_y y \cdot P(Y=y) \\
&= E[X] + E[Y].
\end{aligned}
$$



- Note that we have never used any properties of independence in this proof, and thus linearity of expectation holds for all random variables!
- For continuous random variables, the proof is essentially the same except that the summations are replaced by integrals.

https://brilliant.org/wiki/linearity-of-expectation/

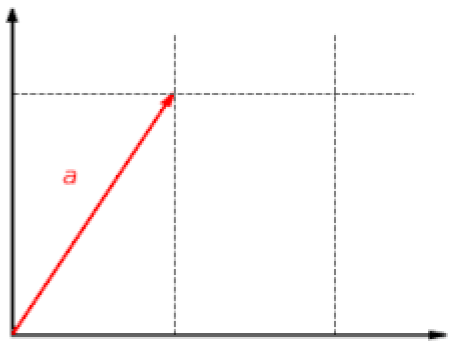
Linear Transformations

https://en.wikipedia.org/wiki/Linear_map

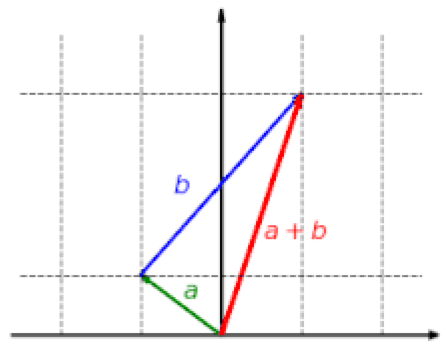
http://mathworld.wolfram.com/LinearTransformation.html

- A **Linear Transformation** (or linear map) is a special type of function where:
  - F(u + v) = F(u) + F(v) and
  - F(*cv*) = *c*F(v) for a constant/scalar *c*.

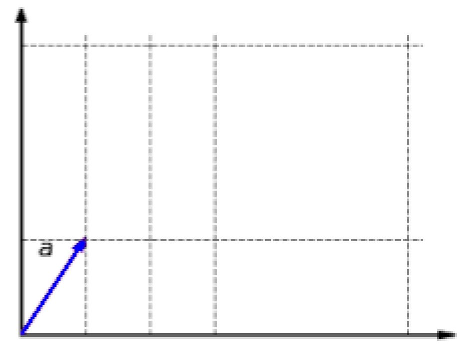
These functions are extensively studied in Linear Algebra (Math 200) and get their name by always mapping a line into a line.



The function $f: \mathbb{R}^2 \to \mathbb{R}^2$ with f(x,y)=(2x,y) is a linear map. This function scales the x component of a vector by the factor 2.



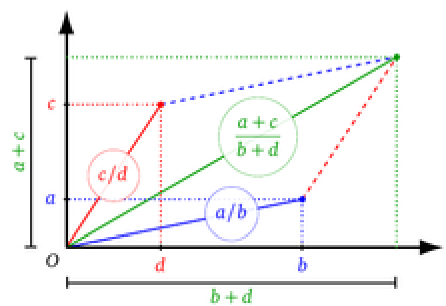
The function is additive: It doesn't matter whether first vectors are added and then mapped or whether they are mapped and finally added: f(u+v)=f(u)+f(v)



The function is homogeneous: It doesn't matter whether a vector is first scaled and then mapped or first mapped and then scaled: f(cv)=cf(v)

Is everything linear?

- Can we apply this procedure universally, just because its easy?
- Lets consider a “freshman sum“ from math students, when requested to add fractions
- $\frac{2}{7}+\frac{3}{5}=\frac{5}{12}$ Right!!
- In reality this “mistake” leads us to interesting area and study of numbers called the Mediant $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$.
- IF we do allow this new interesting relationships can occur

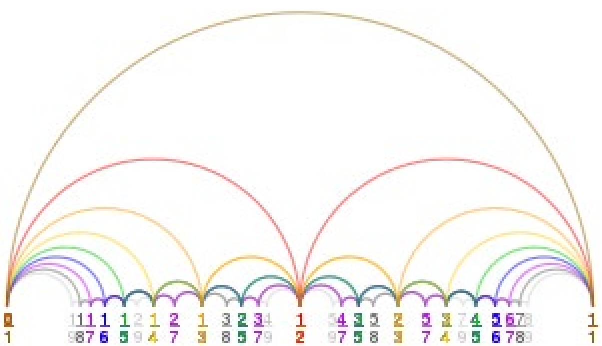


https://www.nctm.org/Publications/Mathematics-Teacher/2016/Vol110/Issue1/mt2016-08-18a/

- https://en.wikipedia.org/wiki/Mediant_(mathematics)

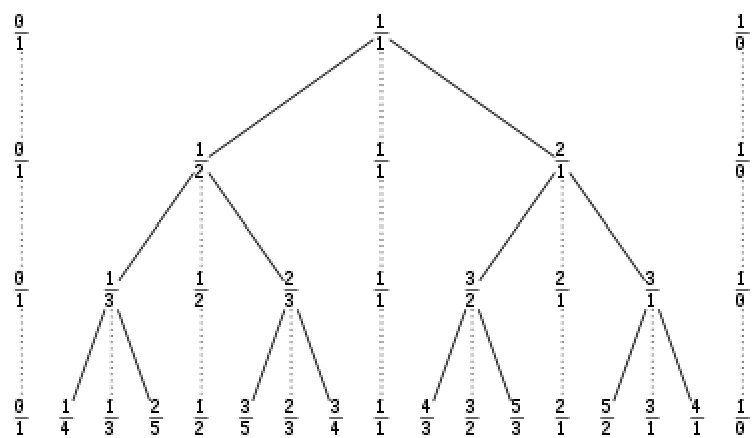
More on mediant

- **Not todays topic**, but want to entice your math curiosity, look at:
- Farey sequences



https://en.wikipedia.org/wiki/Farey_sequence

- Stern-Brocot tree



https://en.wikipedia.org/wiki/Stern%E2%80%93Brocot_tree

Fundamental Tragedy of algebra

- That sinister bug has raised his head,
- And like a germ he is starting to spread.
- Distributing exponents is the sign,
- That this bug is on the climb.
- Look at that student over there,
- Distributing exponents without a care.
- Please listen to your maker,
- Distributing exponents will bring the undertaker.
- Dear Lord please open your gates.
- Being a math student was not his fate.
- Distributing exponents was his only sin.
- But that's enough to do an algebra student in.
- An example, his demise should serve,
- For other students who haven't heard,
- Distributing exponents is a sin.
- It's enough to do an algebra student in.
-
- Donald E. Brook
- Mt. San Antonio College

- $(a+b)^2 \neq a^2+b^2$
-
- $\sqrt{a^2+b^2} \neq a+b$
-
- $(a^2+b^2)^{1/2} \neq a+b$
-
- $a^{-1}+b^{-1} \neq (a+b)^{-1}$
-
- $a^{-1}+b^{-1} \neq -\frac{1}{a+b}$
-
- $\sqrt[3]{a^3+b^3} \neq a+b$
-
- $(a+b)^3 \neq a^3+b^3$
-
- $(a+b)^4 \neq a^4+b^4$
-
- $\left(\sqrt{a}+\sqrt{b}\right)^2 \neq a+b$

φ

Fundamental Tragedy of linearity

- Linear breakdowns in other areas of mathematics:
  - Algebra: $\log_b(x) + \log_b(y) \neq \log_b(x + y)$
  - Recall $\log_b x=y \leftrightarrow b^y=x$
  - Correct identity $\log_b(x) + \log_b(y) = \log_b(xy)$, Proof:
  - Let $b^A=x$ and $b^B=y$, then $\log_b(xy)=\log_b(b^A b^B)=\log_b(b^{A+B})=A+B= \log_b(x) + \log_b(y)$
  - Logic: $\neg(p\land q)\neq \neg p\land\neg q$, likewise $\neg(p\lor q)\neq \neg p\lor\neg q$

p	q	p∧q	¬(p∧q)	¬p∧¬q	¬p∨¬q
F	F	F	T	T	T
F	T	F	T	F	T
T	F	F	T	F	T
T	T	T	F	F	F

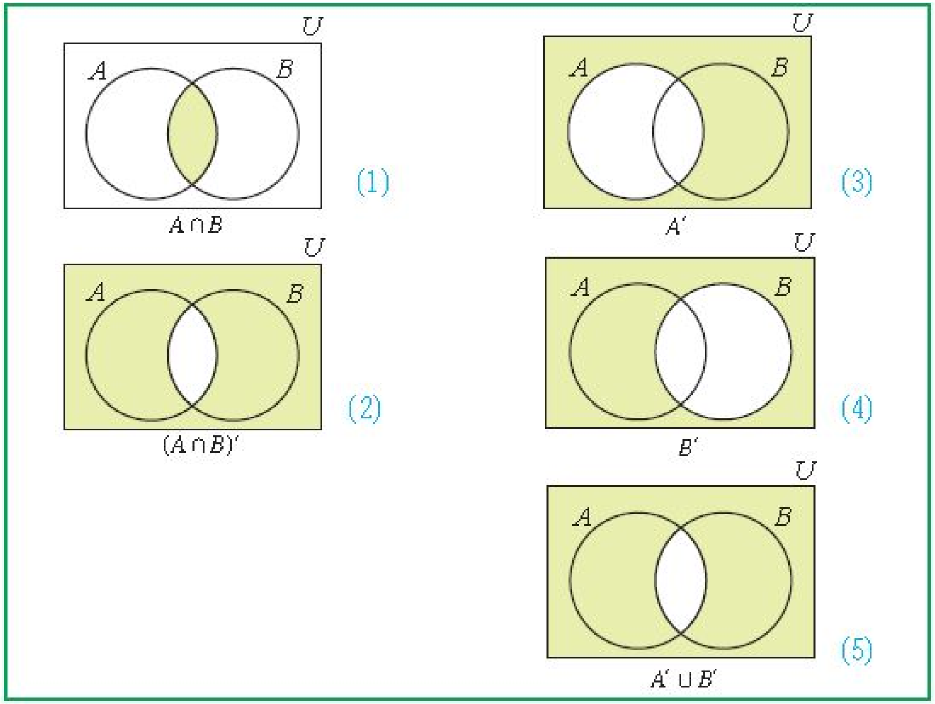
p	q	p∧q	¬(p∨q)	¬p∨¬q	¬p∧¬q
F	F	F	T	T	T
F	T	F	F	T	F
T	F	F	F	T	F
T	T	T	F	F	F

  - Rather by DeMorgans law $\neg(p\land q)= \neg p\lor\neg q$ likewise $\neg(p\lor q)= \neg p\land\neg q$

Fundamental Tragedy of linearity

- Linear breakdowns in other areas of mathematics:
  - Sets: $(A \cap B)' \neq A' \cap B'$ likewise $(A \cup B)' \neq A' \cup B'$

  - Rather by DeMorgans law $(A \cap B)' \neq A' \cup B'$, proof below.
  - Likewise $(A \cup B)' = A' \cap B'$. Try proving this second version.

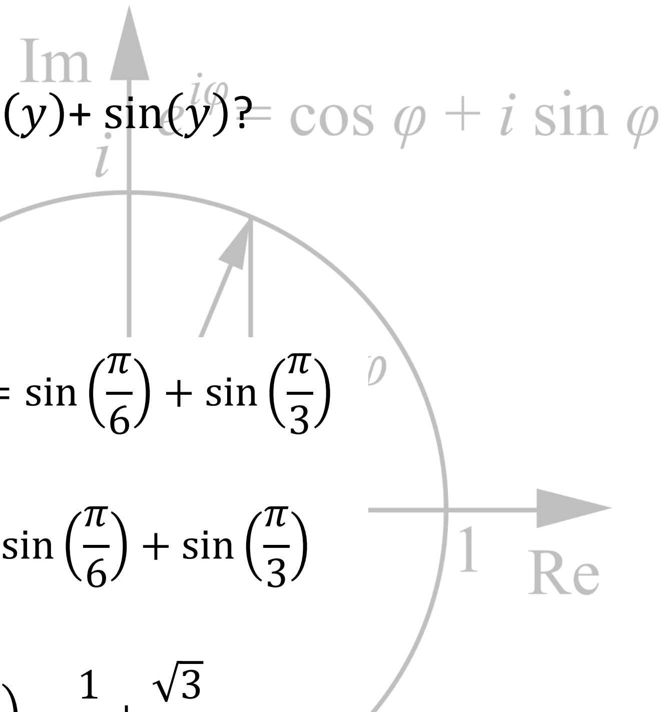


Attempt at Linearity

What should $\cos(x+y)$ and $\sin(x+y)$ be?

Do these trigonometric functions behave linearly?

Is $\cos(x+y) = \cos(x) + \cos(y)$ and $\sin(x+y) = \sin(y) + \sin(y)$?

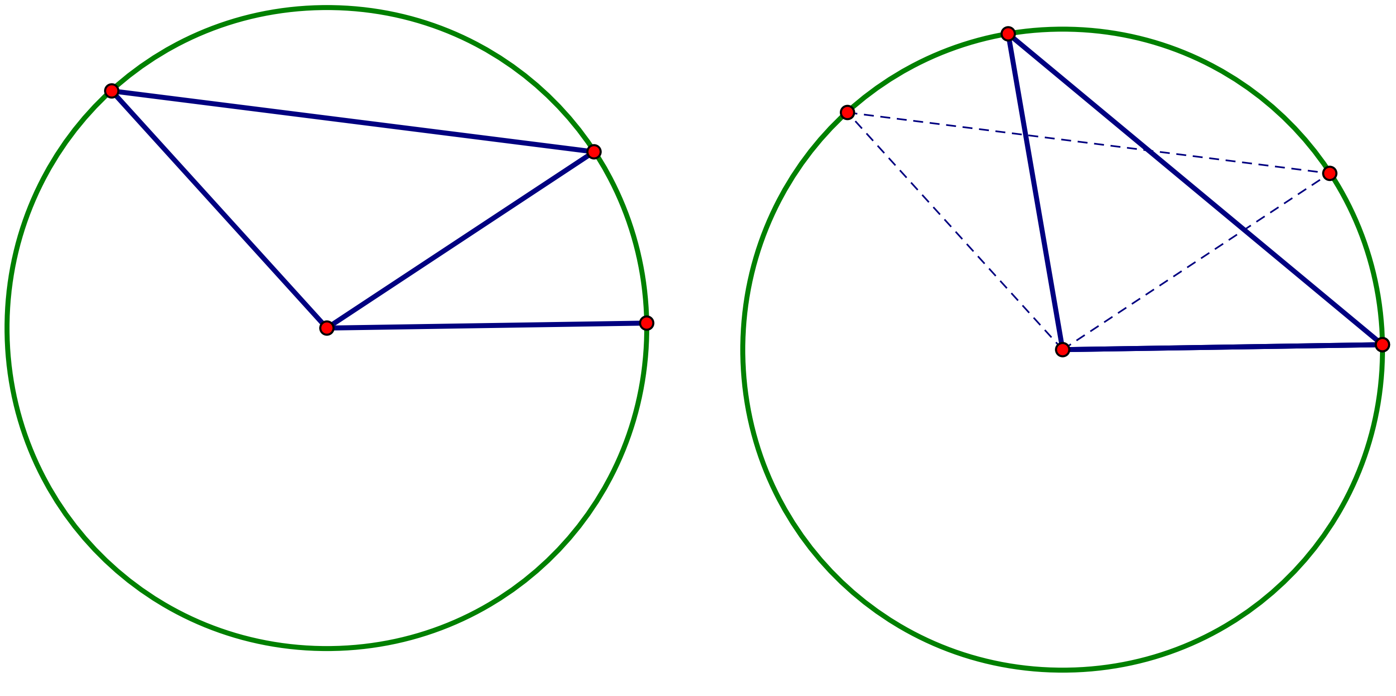


Try with some known values:

$\cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)$	$\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)$
$\cos\left(\frac{3\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)$	$\sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)$
$\cos\left(\frac{\pi}{2}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}$	$\sin\left(\frac{\pi}{2}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}$
$0 = \frac{1+\sqrt{3}}{2}$?	$1 = \frac{1+\sqrt{3}}{2}$?

So that failed… let's try distance!

Find $\cos(x - y)$ based on the unit circle.



So that failed… let's try distance!

Find $\cos(x - y)$ based on the unit circle. Label the coordinates of each point.

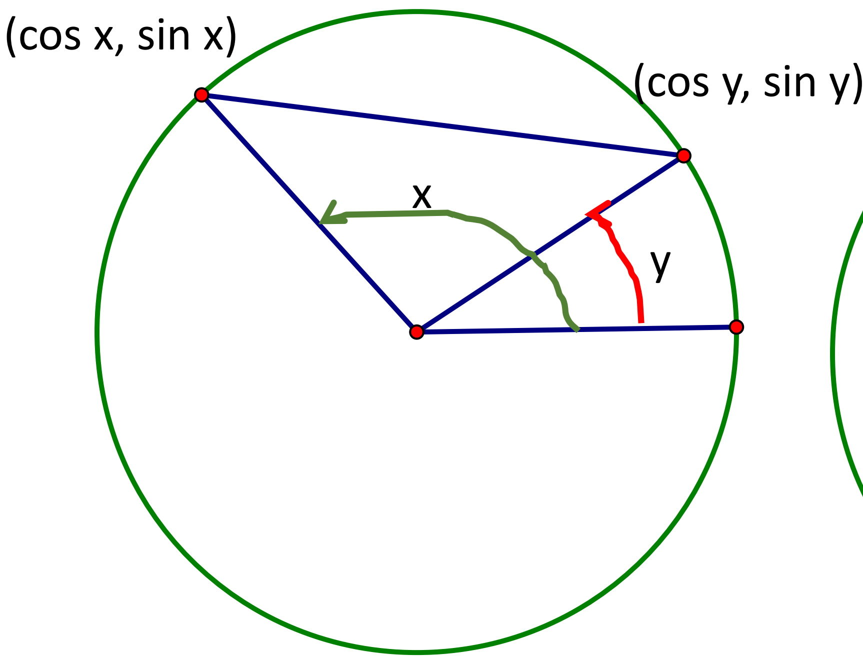


Figure 1

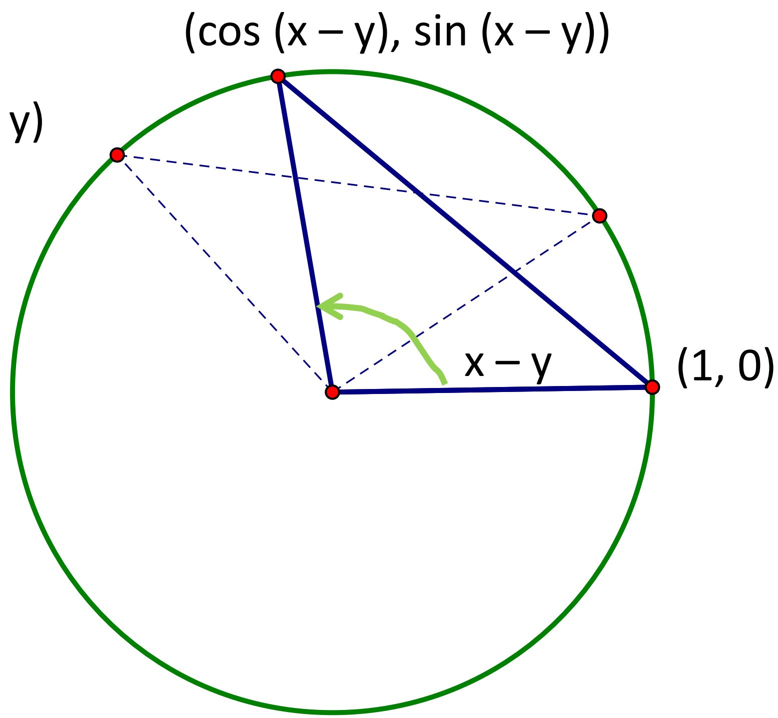
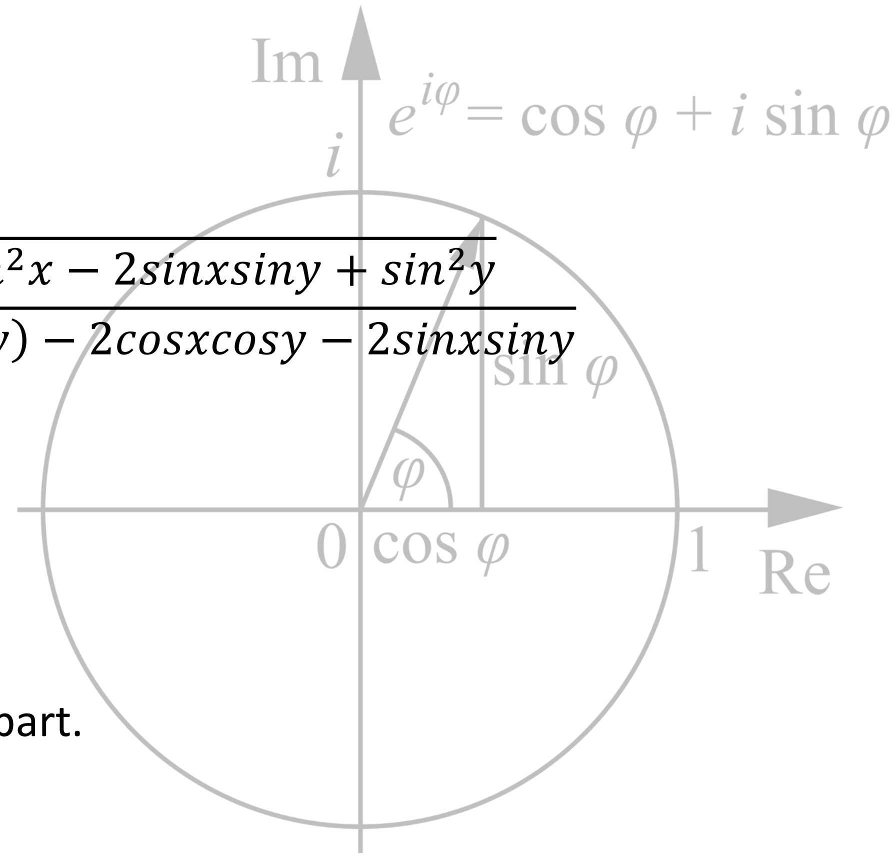


Figure 2

Find $\cos(x-y)$ based on the unit circle.

Distance between the two labeled points in Figure 1.



$$d = \sqrt{(cosx - cosy)^2 + (sinx - siny)^2}$$
$$d = \sqrt{cos^2x - 2cosxcosy + cos^2y + sin^2x - 2sinxsiny + sin^2y}$$
$$d = \sqrt{(cos^2x + sin^2x) + (cos^2y + sin^2y) - 2cosxcosy - 2sinxsiny}$$
$$d = \sqrt{1 + 1 - 2cosxcosy - 2sinxsiny}$$
$$d = \sqrt{2 - 2cosxcosy - 2sinxsiny}$$
$$d = \sqrt{2 - 2(cosxcosy + sinxsiny)}$$

Keep this in mind as we move to the next part.

Find $\cos(x-y)$ based on the unit circle.

Distance between the two labeled points in Figure 2.

$$d = \sqrt{(cos(x-y)-1)^2 + (sin(x-y)-0)^2}$$
$$d = \sqrt{cos^2(x-y) - 2cos(x-y) + 1 + sin^2(x-y)}$$
$$d = \sqrt{\left(cos^2(x-y) + sin^2(x-y)\right) + 1 - 2cos(x-y)}$$
$$d = \sqrt{1+1-2cos(x-y)} = \sqrt{2-2cos(x-y)}$$

Now compare this to the previous distance: $d = \sqrt{2-2(cosxcosy + sinxsiny)}$

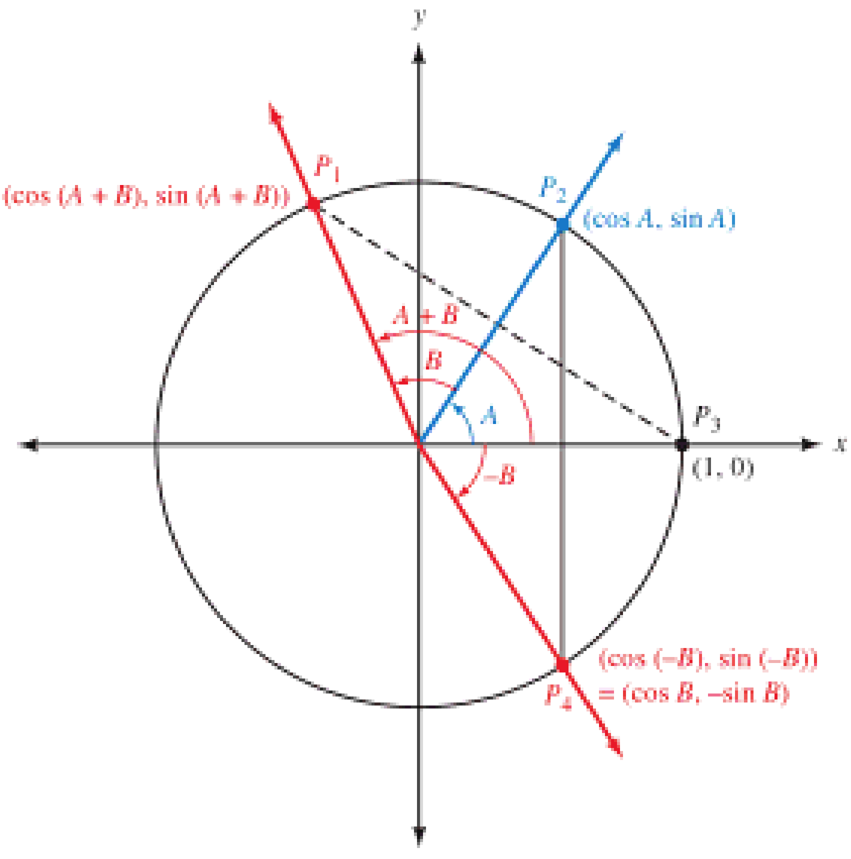
Since the distances must be the same, we conclude that:
$\boldsymbol{cos(x-y) = cosxcosy + sinxsiny}$

Want $sin(x-y)$, then just check the complement (cofunction Identity):

$$\boldsymbol{sin(x-y)} = cos\left(\frac{\pi}{2} - (x-y)\right) = cos\left(\left(\frac{\pi}{2} - x\right) - (-y)\right) =$$
$$\cos\left(\frac{\pi}{2} - x\right)\cos(-y) + \sin\left(\frac{\pi}{2} - x\right)\sin(-y) = \boldsymbol{sinxcosy - cosxsiny}$$

What about the other ones?

You can use a similar picture to graph addition, but in this case, you'll need to think of a clockwise rotation of an angle, so one angle will be negative.

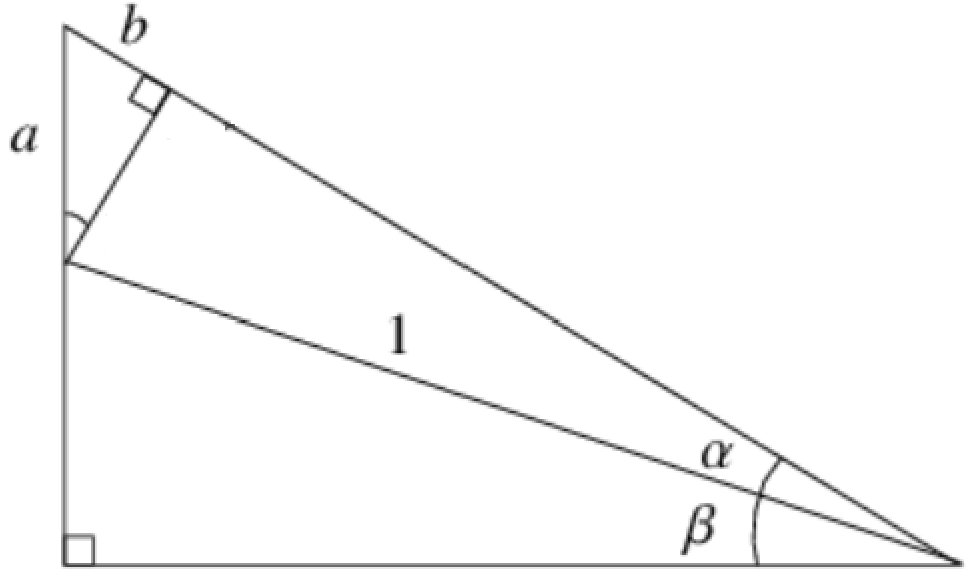


Summary:

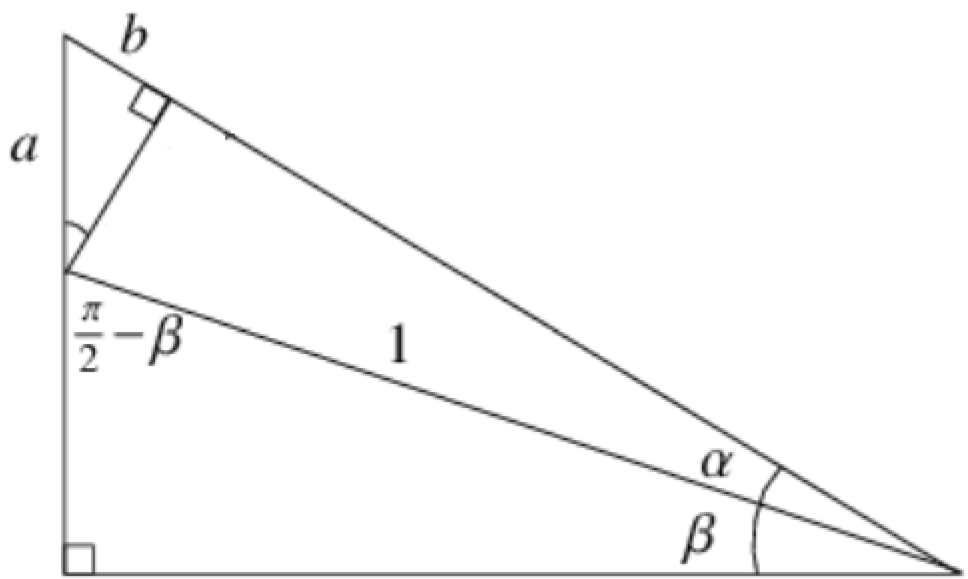
$$cos(x \pm y) = cosxcosy \mp sinxsiny$$

$$sin(x \pm y) = sinxcosy \pm cosxsiny$$

Deriving sum identity using SOHCAHTOA, and without the Unit circle.



http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html



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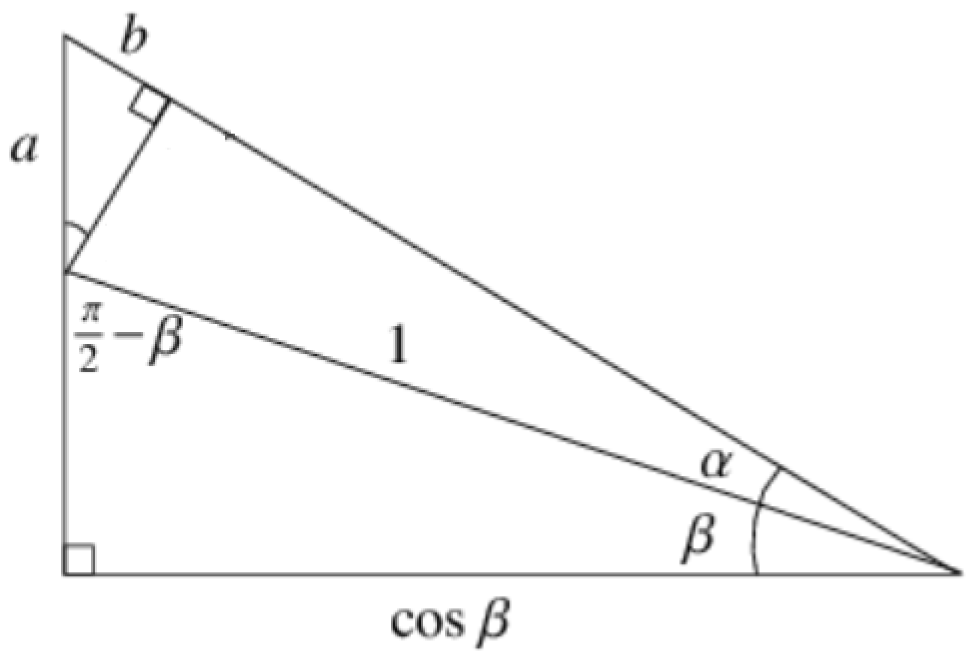
Deriving sum identity using SOHCAHTOA, and without the Unit circle.



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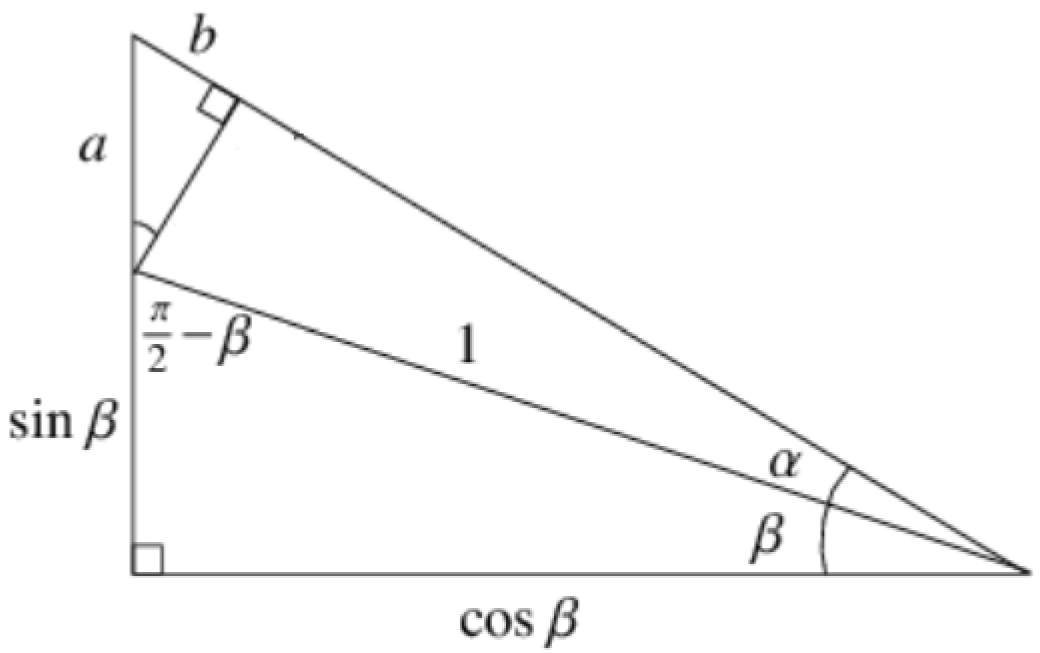
Deriving sum identity using SOHCAHTOA, and without the Unit circle.



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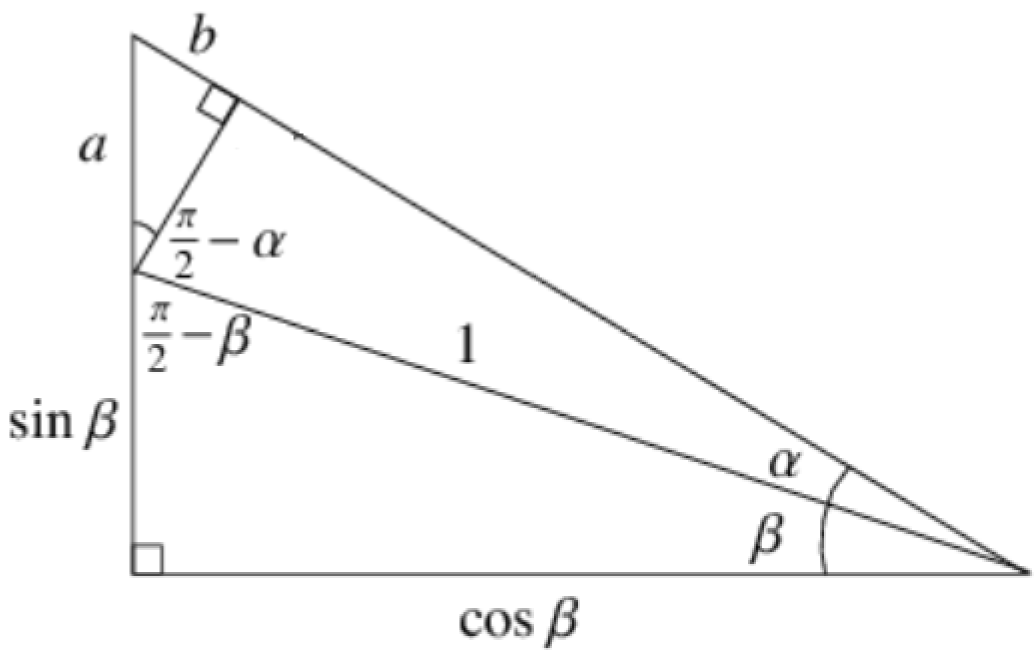
Deriving sum identity using SOHCAHTOA, and without the Unit circle.



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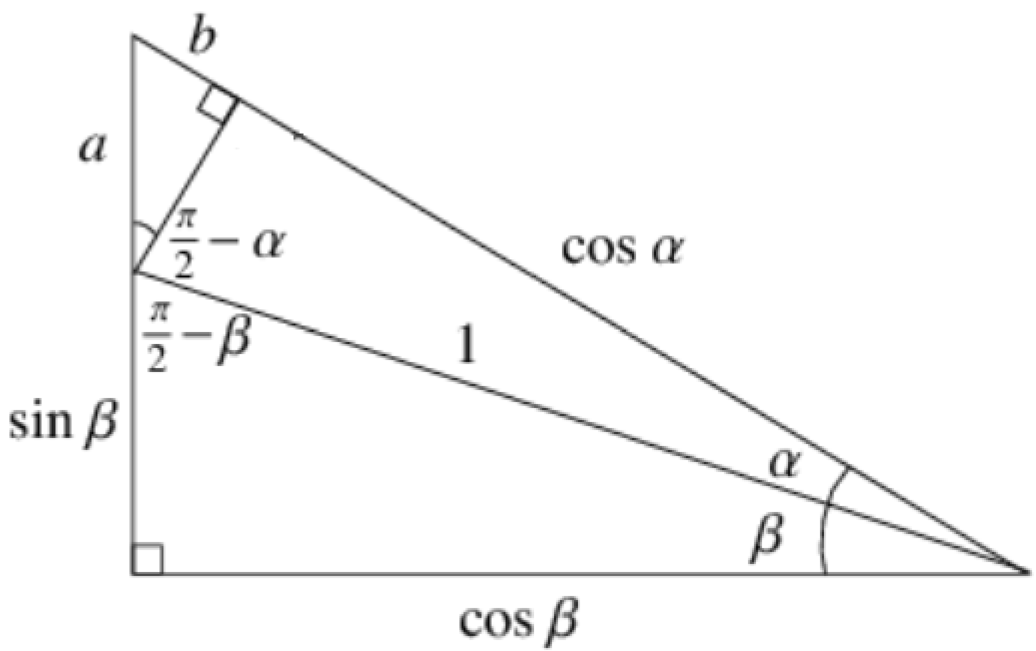
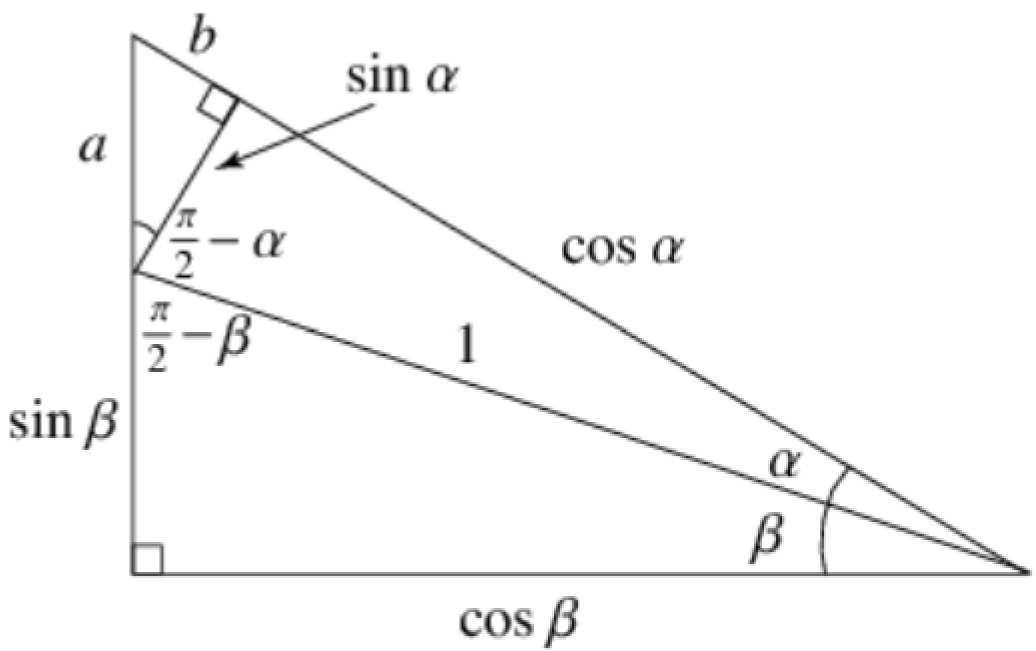
Deriving sum identity using SOHCAHTOA, and without the Unit circle.



http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html

Deriving sum identity using SOHCAHTOA, and without the Unit circle.

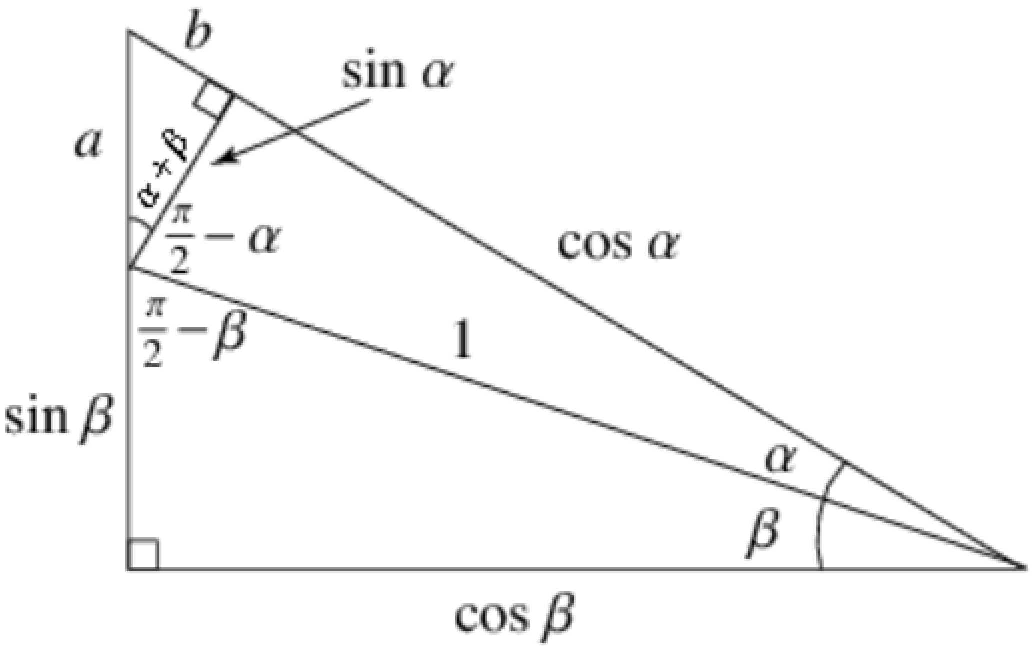


http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html

Deriving sum identity using SOHCAHTOA, and without the Unit circle.

- Consider the small right triangle in the figure above, which gives

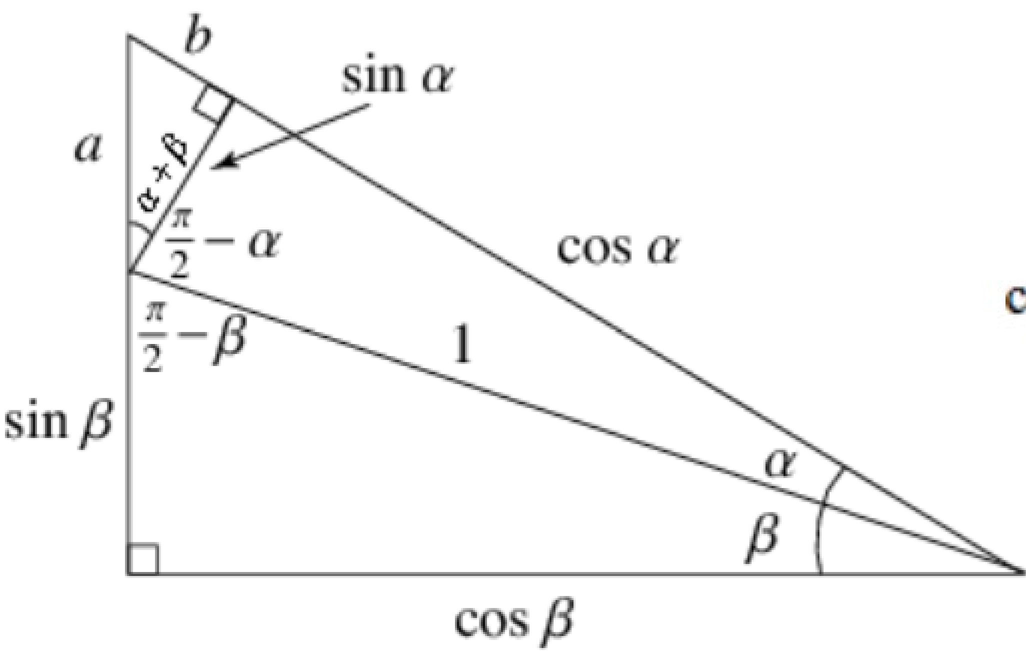
$$a = \frac{\sin\alpha}{\cos(\alpha+\beta)}$$
$$b = \sin\alpha\tan(\alpha+\beta).$$



http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html

Deriving sum identity using SOHCAHTOA, and without the Unit circle.

- Now, the usual trigonometric definitions applied to the large right triangle give



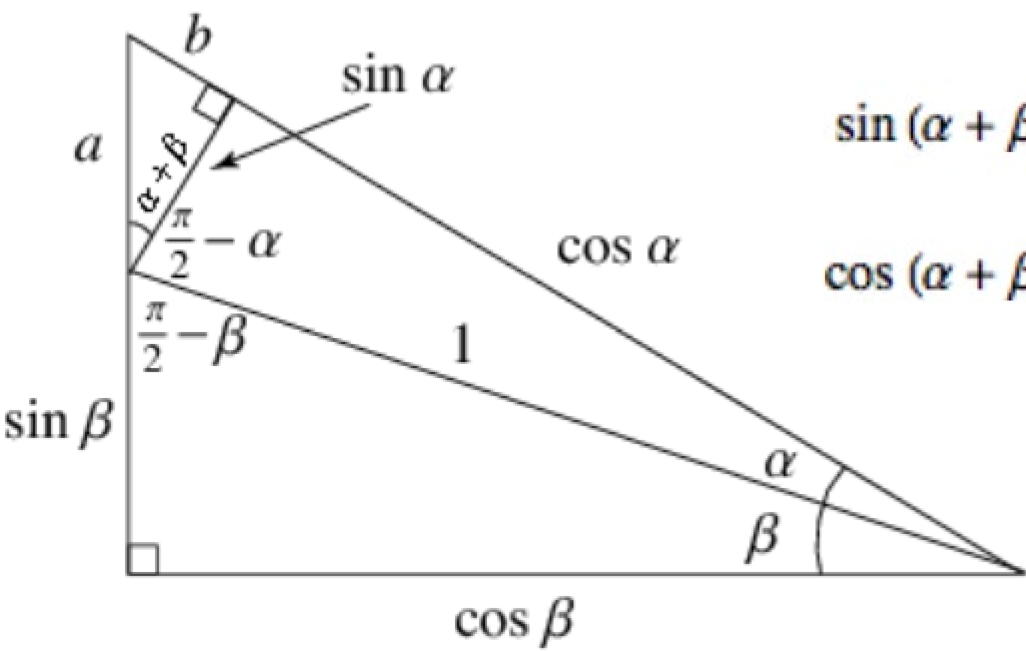
$$\sin(\alpha+\beta) = \frac{\sin\beta + a}{\cos\alpha + b} = \frac{\sin\beta + \frac{\sin\alpha}{\cos(\alpha+\beta)}}{\cos\alpha + \sin\alpha\,\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}}$$

$$\cos(\alpha+\beta) = \frac{\cos\beta}{\cos\alpha + b} = \frac{\cos\beta}{\cos\alpha + \sin\alpha\,\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}}.$$

http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html

Deriving sum identity using SOHCAHTOA, and without the Unit circle.

- Solving these two equations simultaneously for the variables sin(α+β) and cos(α +β) then immediately gives



$$\sin(\alpha+\beta) = \frac{\cos\alpha\sin\alpha + \cos\beta\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

$$\cos(\alpha+\beta) = \frac{\cos^2\beta - \sin^2\alpha}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}.$$

http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html

Deriving sum identity using SOHCAHTOA, and without the Unit circle.

- These can be put into the familiar forms with the aid of the trigonometric identities

$$(\cos\alpha\cos\beta + \sin\alpha\sin\beta)(\sin\alpha\cos\beta + \sin\beta\cos\alpha) = \cos\beta\sin\beta + \cos\alpha\sin\alpha$$

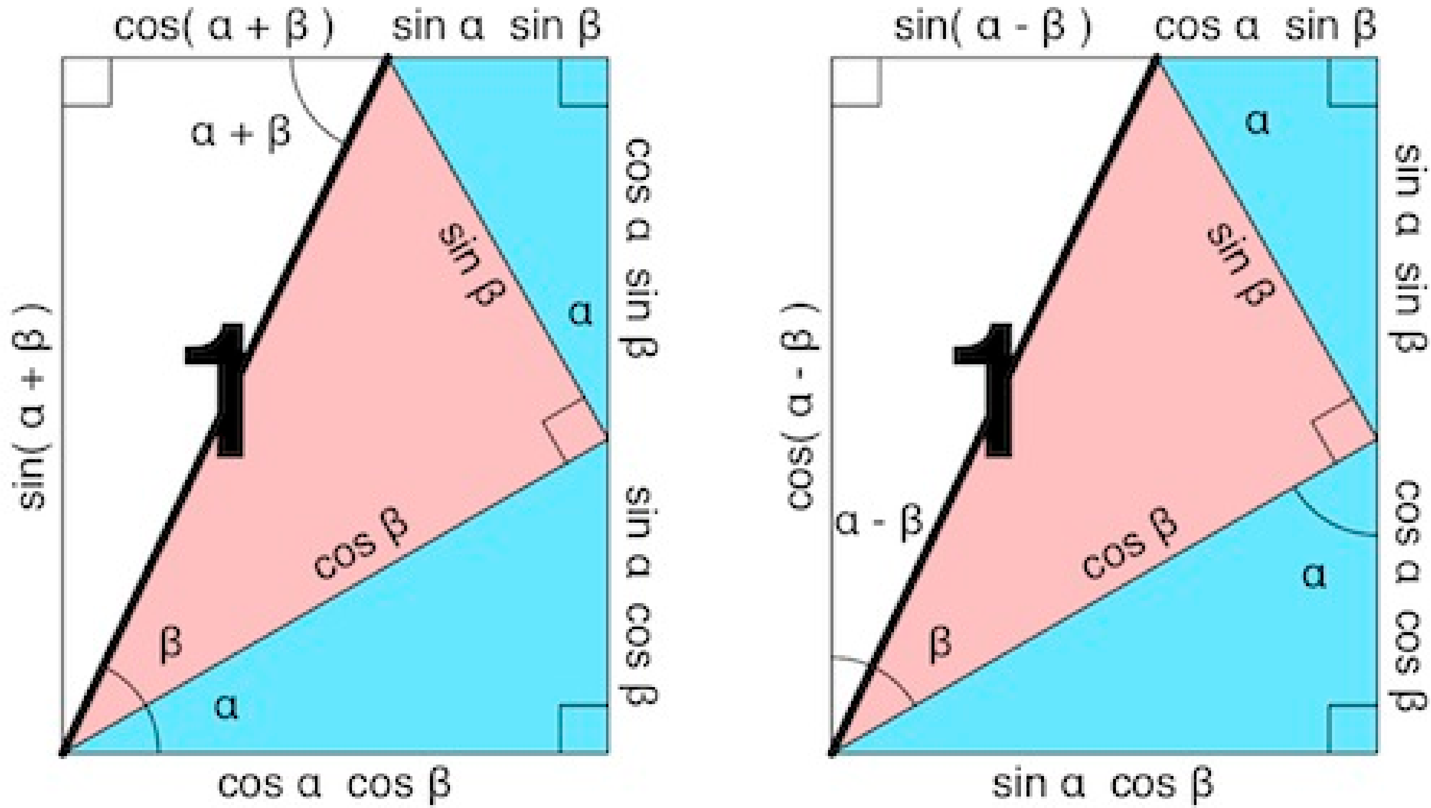
$$\begin{aligned}(\cos\alpha\cos\beta + \sin\alpha\sin\beta)(\cos\alpha\cos\beta - \sin\alpha\sin\beta) &= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta \\ &= 1 - \sin^2\alpha - \sin^2\beta \\ &= \cos^2\alpha - \sin^2\beta \\ &= \cos^2\beta - \sin^2\alpha,\end{aligned}$$

- which can be verified by direct multiplication.
- Plugging these back to the equations in the previous slide gives:

$$\begin{aligned}\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta,\end{aligned}$$

http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html

Other pictures



$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

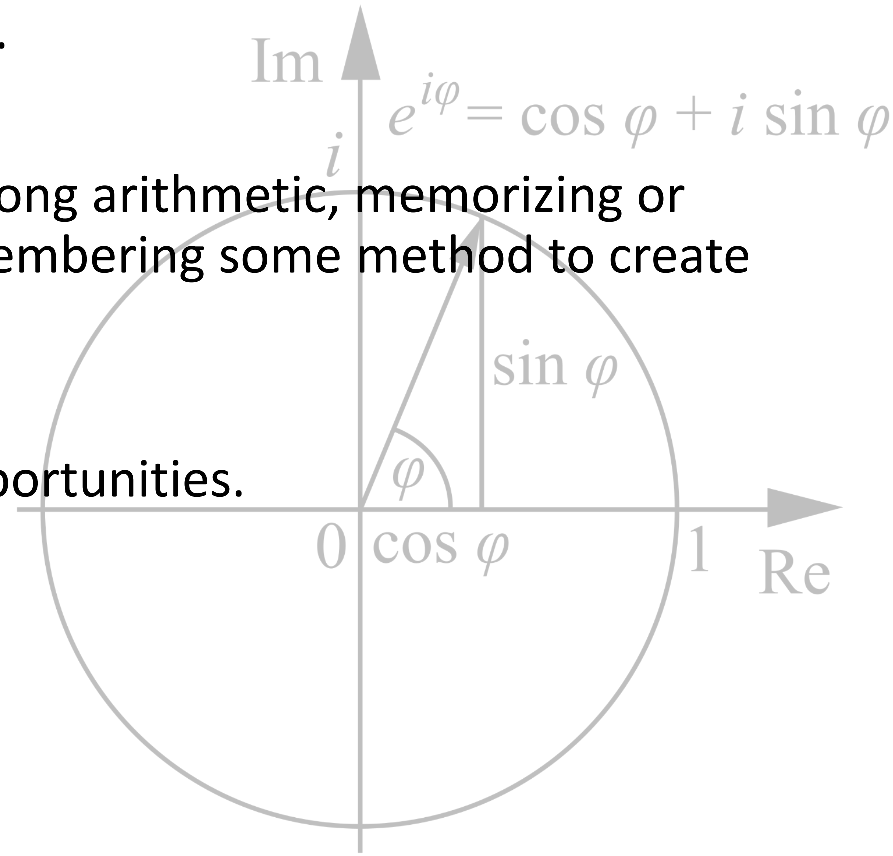
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

My eyes hurt

- Is there another way, that does not involve the unit circle with distance formula or ‘weird’ pictures.

- Is there a way that doesn’t require long arithmetic, memorizing or remembering some picture, or remembering some method to create a picture that would work?

- Lets enter an imaginary word of opportunities.

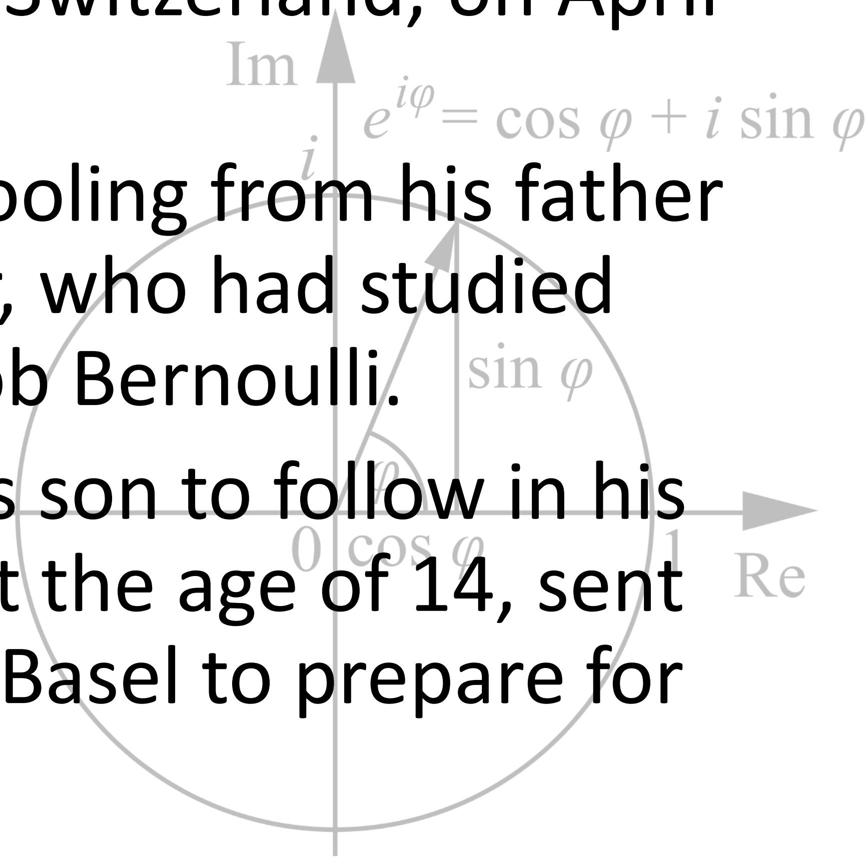


Leonhard Euler [1707-1783]

- Euler is considered the most **prolific** mathematician in history. (What about Erdős?)
- His contemporaries called him “**analysis incarnate**.”
- *“He calculated without effort, just as men breathe or as eagles sustain themselves in the air.”*

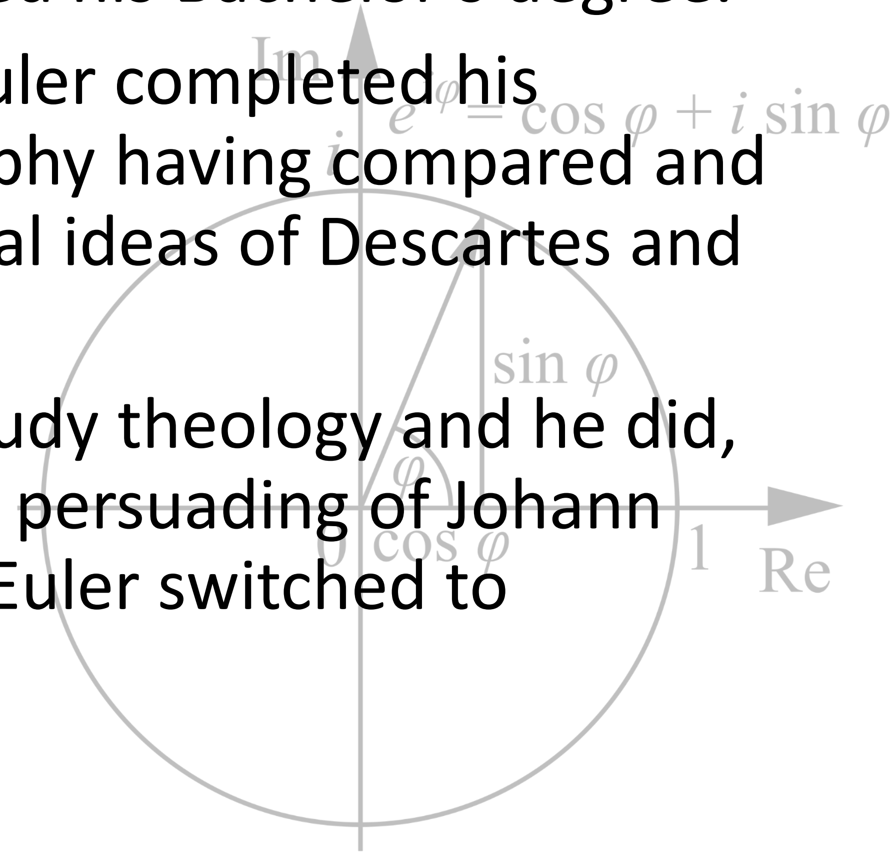
Leonhard Euler [1707-1783]

- Euler was born in Basel, Switzerland, on April 15, 1707.
- He received his first schooling from his father Paul, a Calvinist minister, who had studied mathematics under Jacob Bernoulli.
- Euler's father wanted his son to follow in his footsteps and, in 1720 at the age of 14, sent him to the University of Basel to prepare for the ministry.



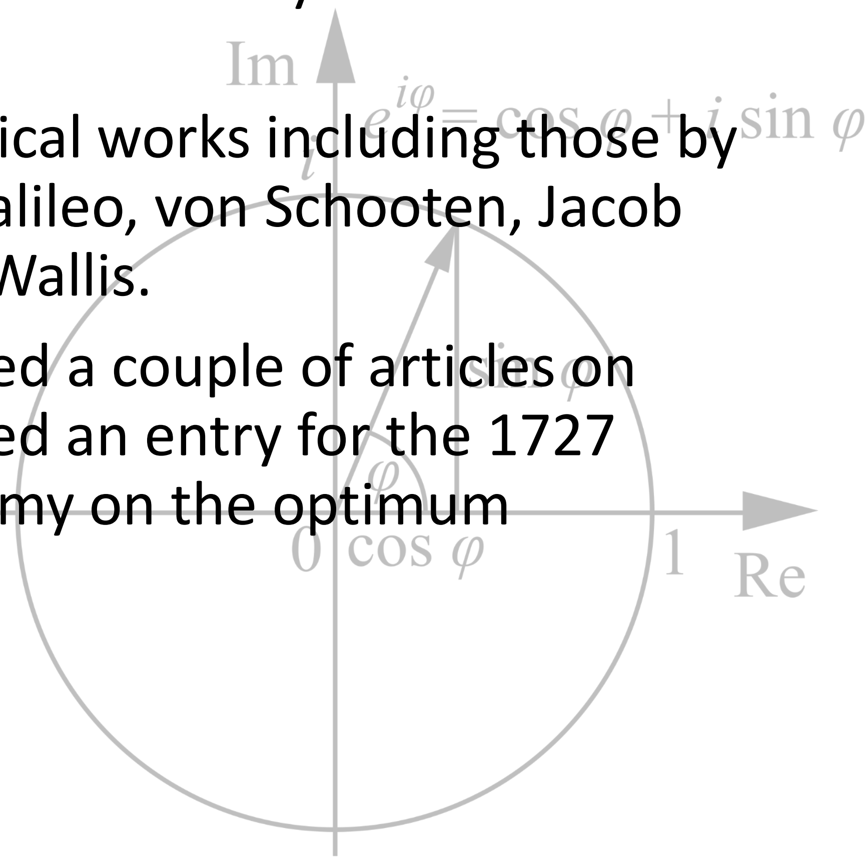
Leonhard Euler [1707-1783]

- At the age of 15, he received his Bachelor’s degree.
- In 1723 at the age of 16, Euler completed his Master's degree in philosophy having compared and contrasted the philosophical ideas of Descartes and Newton.
- His father demanded he study theology and he did, but eventually through the persuading of Johann Bernoulli, Jacob’s brother, Euler switched to mathematics.



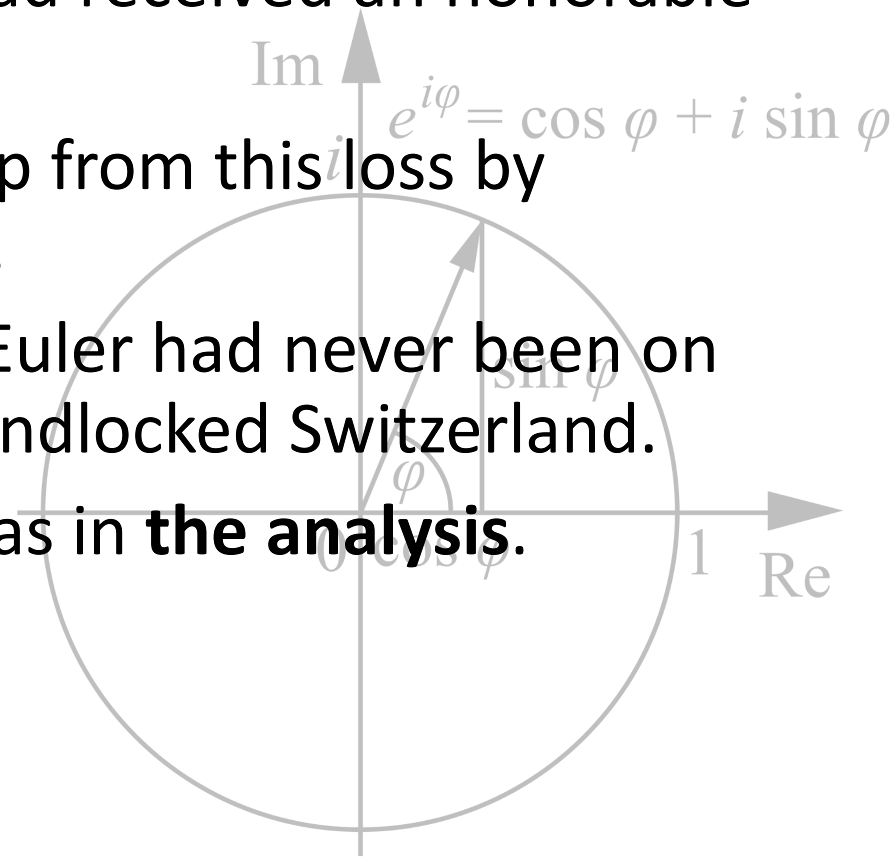
Leonhard Euler [1707-1783]

- Euler completed his studies at the University of Basel in 1726.
- He had studied many mathematical works including those by Varignon, Descartes, Newton, Galileo, von Schooten, Jacob Bernoulli, Hermann, Taylor and Wallis.
- By 1727, he had already published a couple of articles on isochronous curves and submitted an entry for the 1727 Grand Prize of the French Academy on the optimum placement of masts on a ship.



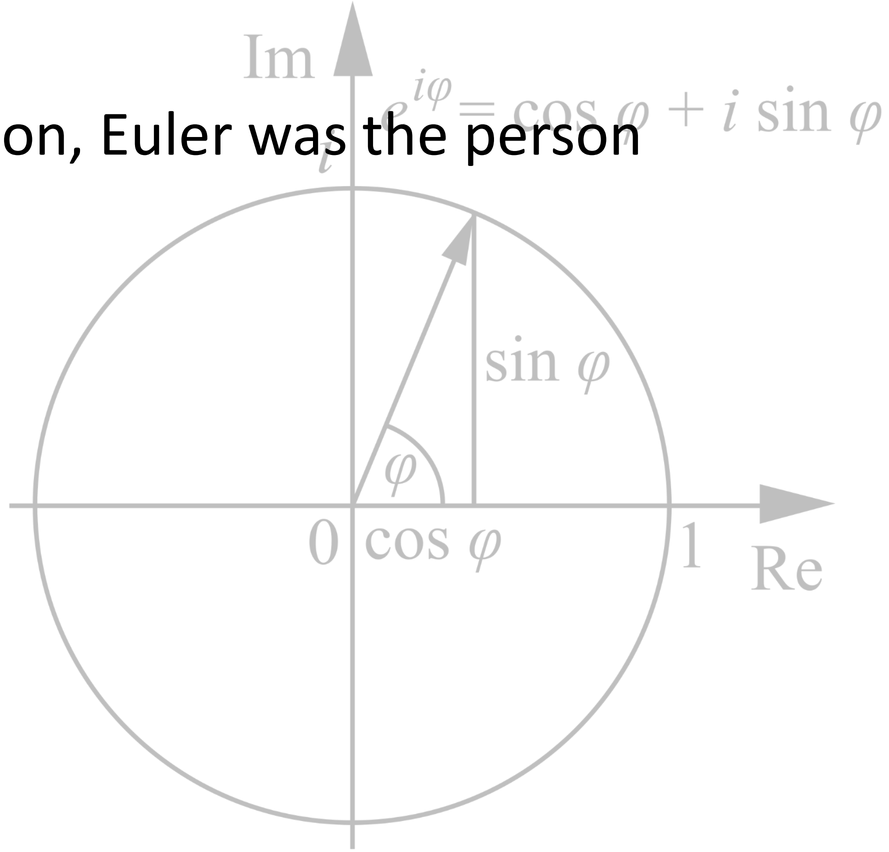
Leonhard Euler [1707-1783]

- Euler did not win but instead received an honorable mention.
- He eventually would recoup from this loss by winning the prize 12 times.
- What is interesting is that Euler had never been on a ship having come from landlocked Switzerland.
- The strength of his work was in **the analysis**.



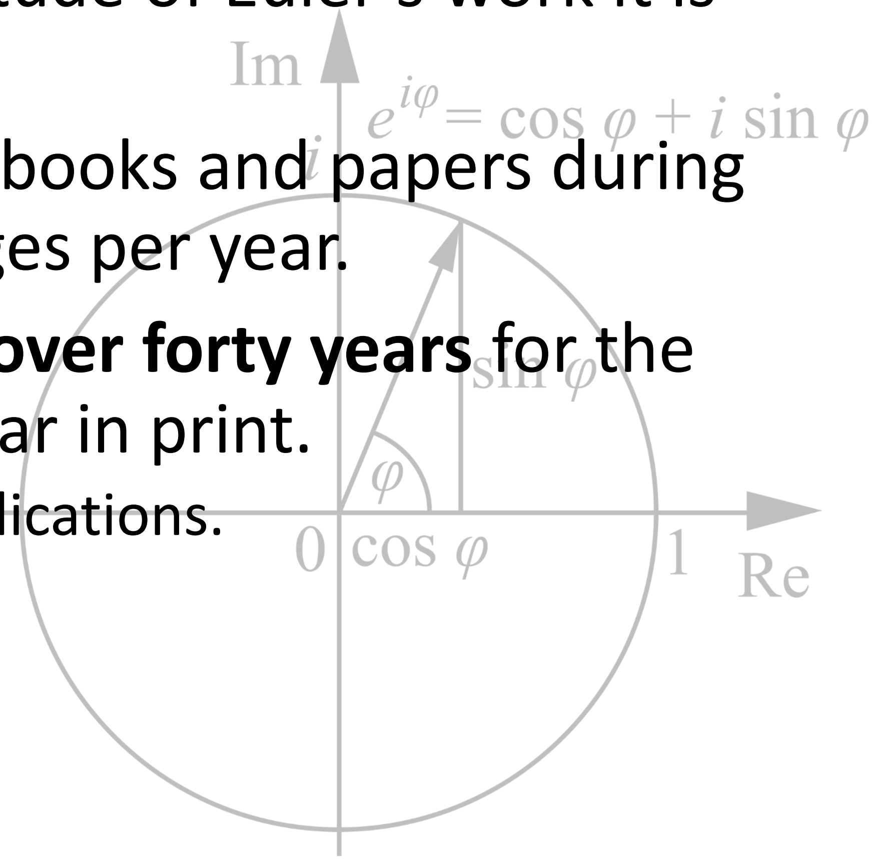
Leonhard Euler [1707-1783]

- Euler was in a sense the creator of modern mathematical expression.
- In terms of mathematical notation, Euler was the person who gave us:
  - π for pi
  - i for $\sqrt{-1}$
  - Δy for the change in y
  - $f(x)$ for a function
  - Σ for summation



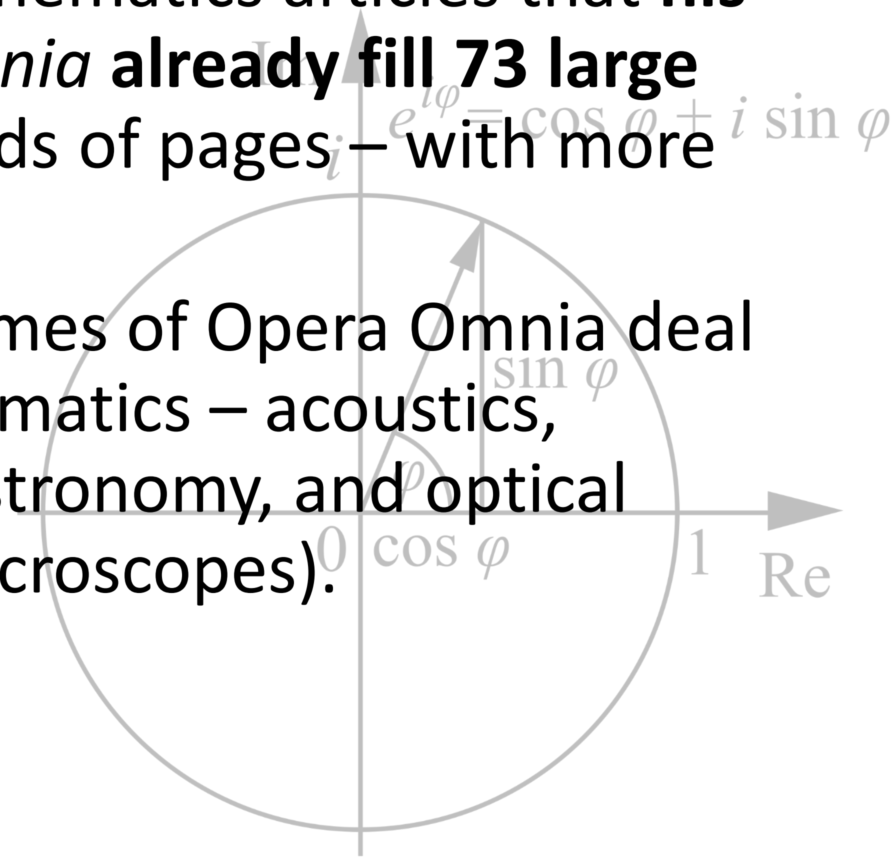
Leonhard Euler [1707-1783]

- To get an idea of the magnitude of Euler’s work it is worth noting that:
- Euler wrote more than 500 books and papers during his lifetime – about 800 pages per year.
- After Euler’s death, **it took over forty years** for the backlog of his work to appear in print.
    - Approximately 400 more publications.



Leonhard Euler [1707-1783]

- He published so many mathematics articles that **his collected works** *Opera Omnia* **already fill 73 large volumes** – tens of thousands of pages – with more volumes still to come.
- More than half of the volumes of Opera Omnia deal with applications of mathematics – acoustics, engineering, mechanics, astronomy, and optical devices (telescopes and microscopes).

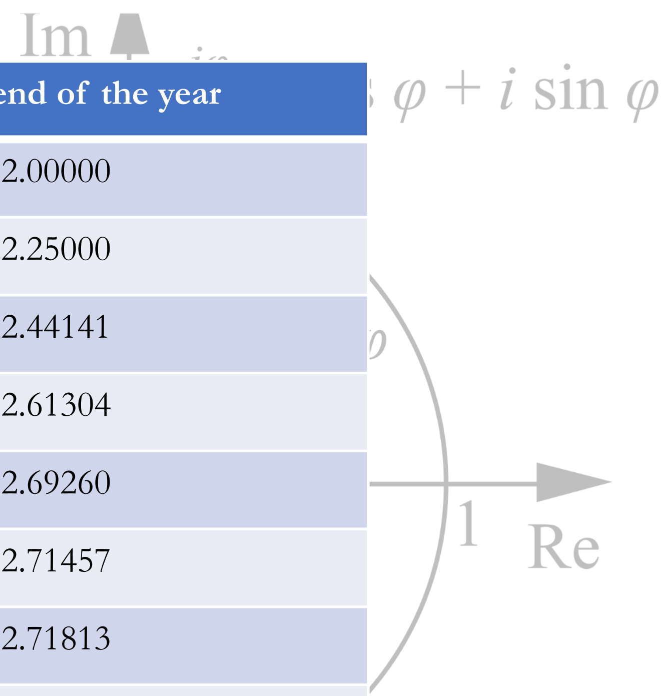


The number e and compound interest

- Invest \$1
- Interest rate 100%

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

"Compound interest is the most powerful force in the universe."
Albert Einstein.

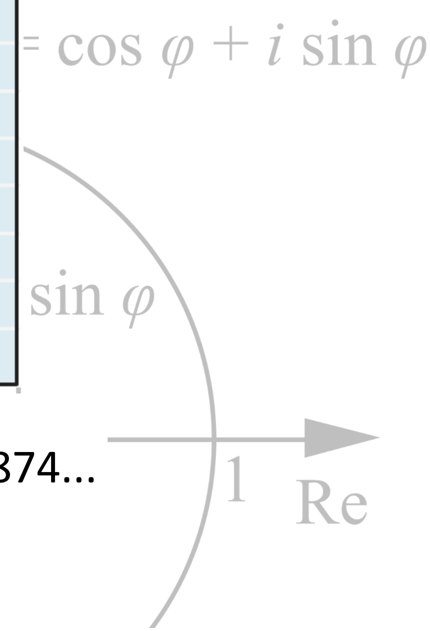


Interest applied each	Sum at end of the year
Year	\$2.00000
Half-year	\$2.25000
Quarter	\$2.44141
Month	\$2.61304
Week	\$2.69260
Day	\$2.71457
Hour	\$2.71813
Minute	\$2.71828
Second	\$2.71828

What about 'continuous' compounding?

The number e as a limit

n	$\frac{1}{n}$	$1+\frac{1}{n}$	$\left(1+\frac{1}{n}\right)^n$
1	1	2	2
2	0.5	1.5	2.25
5	0.2	1.2	2.48832
10	0.1	1.1	2.59374246
100	0.01	1.01	2.704813829
1,000	0.001	1.001	2.716923932
10,000	0.0001	1.0001	2.718145927
100,000	0.00001	1.00001	2.718268237
1,000,000	0.000001	1.000001	2.718280469
1,000,000,000	10^{-9}	$1+10^{-9}$	2.718281827



$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182845904523536028 74\ldots$$

So for continuous compounding, let x=n/r we would get

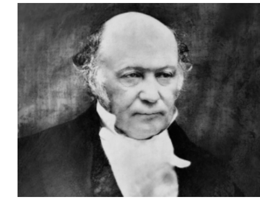
$$A = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = P\cdot\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{xrt} = P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{rt} = Pe^{rt}$$

https://en.wikipedia.org/wiki/E_(mathematical_constant)

Euler on complex numbers

Of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible.

Complex Numbers
William Rowan Hamilton 1805 - 1865



We **define** a complex number as a **pair** (a, b) of real numbers.

https://www.quantamagazine.org/the-imaginary-numbers-at-the-edge-of-reality-20181025/
http://www.gresham.ac.uk/lectures-and-events/eulers-exponentials

Complex Numbers
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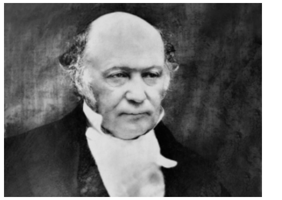

We **define** a complex number as a **pair** (a, b) of real numbers.

They are **added** as follows: $(a, b) + (c, d) = (a + c, b + d)$;

$$(1, 2) + (3, 4) = (4, 6)$$

https://www.quantamagazine.org/the-imaginary-numbers-at-the-edge-of-reality-20181025/
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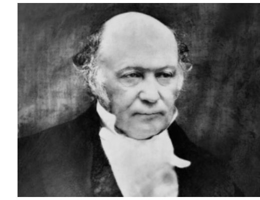

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They are **added** as follows: $(a, b) + (c, d) = (a + c, b + d)$;

They are **multiplied** as follows: $(a, b) \text{ x } (c, d) = (ac - bd, ad + bc)$;

$$(1, 2) \times (3, 4) = (3 - 8, 4 + 6) = (-5, 10)$$

https://www.quantamagazine.org/the-imaginary-numbers-at-the-edge-of-reality-20181025/
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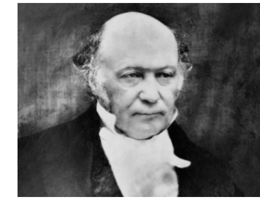

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They are **multiplied** as follows: $(a, b) \times (c, d) = (ac - bd, ad + bc)$;

The pair $(a, 0)$ then **corresponds** to the real number a

the pair $(0, 1)$ **corresponds** to the imaginary number i

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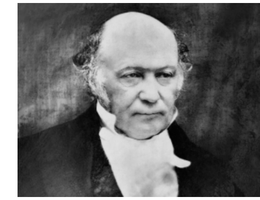

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The pair $(a, 0)$ then **corresponds** to the real number a

the pair $(0, 1)$ **corresponds** to the imaginary number i

Then $(0, 1) \times (0, 1) = (-1, 0)$,

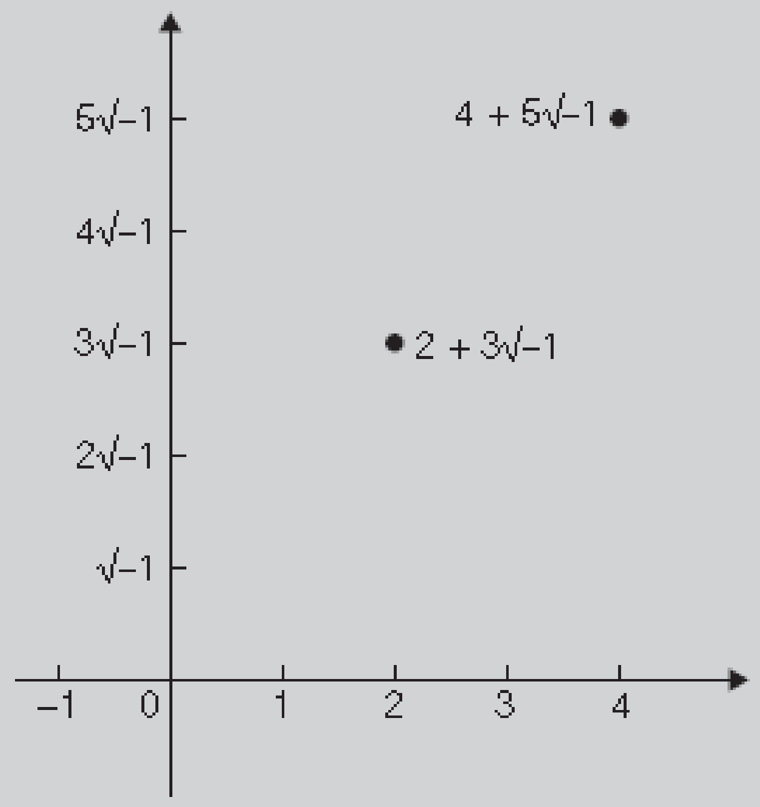
which **corresponds** to the relation

$i \times i = i^2 = -1$.

Representing Complex numbers geometrically

Caspar Wessel in 1799

In this representation, called the complex plane, two axes are drawn at right angles – the real axis and the imaginary axis – and the complex number $a + b\sqrt{-1}$ is represented by the point at a distance a in the direction of the real axis and at height b in the direction of the imaginary axis.



Representing complex numbers

φ

https://en.wikipedia.org/wiki/Complex_number

Four Special Number Systems

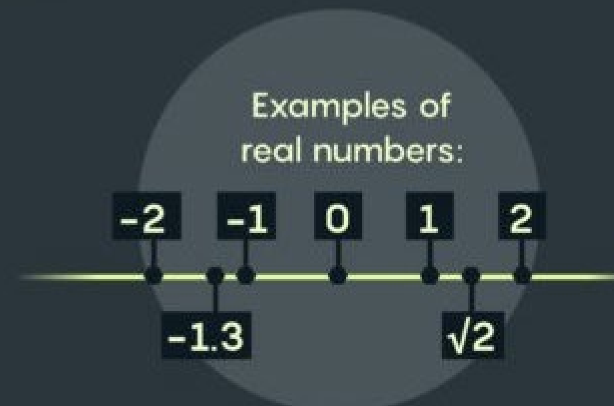
When you add, subtract, multiply or divide the "real numbers" used in everyday life, you always get another real number. Three generalizations of the real numbers also behave in this way. Many physicists believe that all four of these "division algebras" underlie the laws of physics.

ℝ Real numbers

All the numbers on (1-D) number line.

One defining characteristic of reals is that their square is never negative.

e.g. $2^2 = 4$ and also $(-2)^2 = 4$



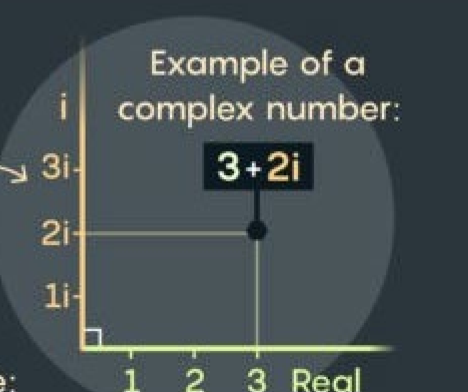
ℂ Complex numbers

Reals used in conjunction with an unconventional "imaginary" unit called i.

One defining characteristic of i is that its square is negative.

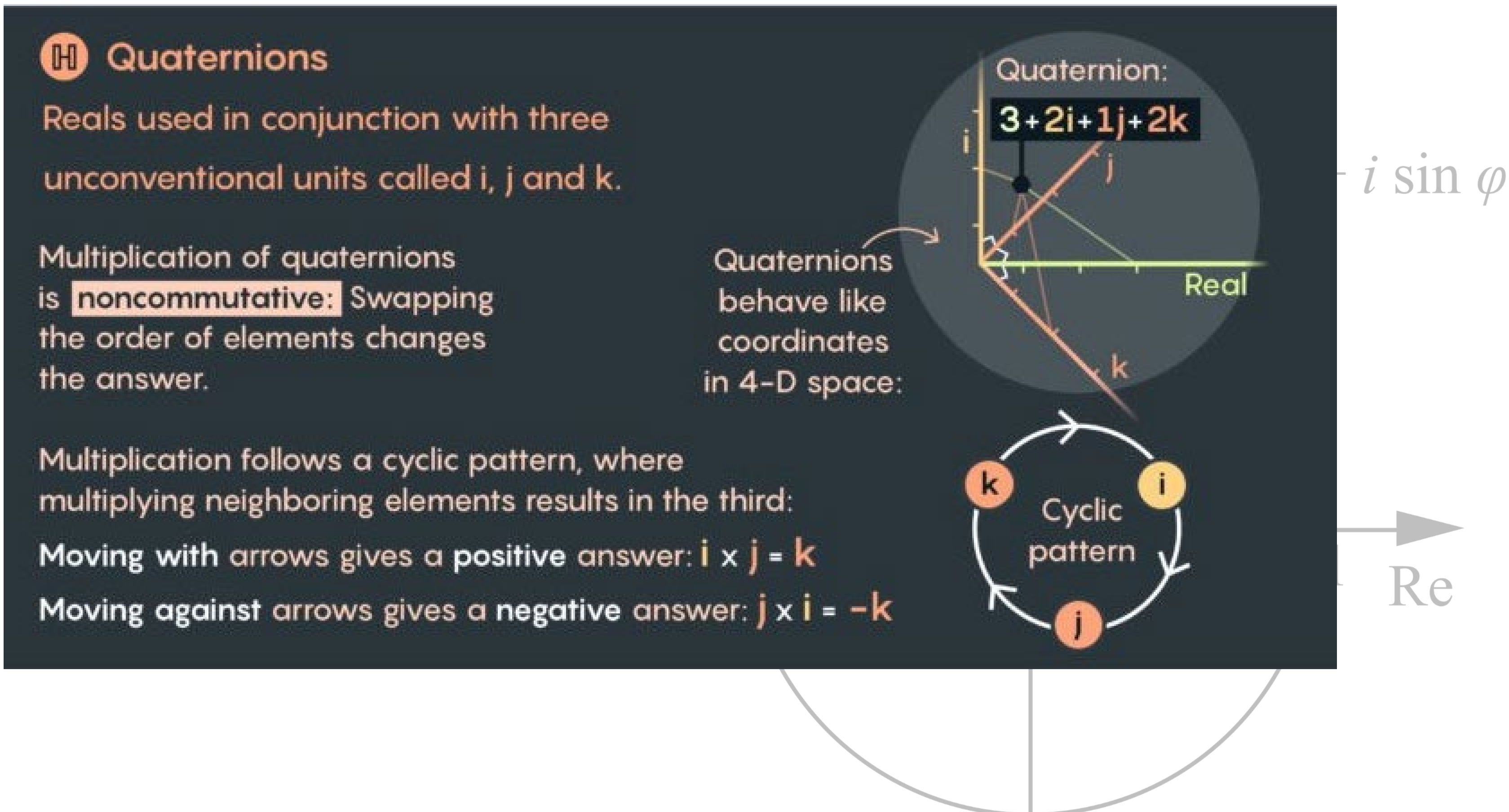
ie. $i^2 = -1$

Complex numbers behave like coordinates on a 2-D plane:



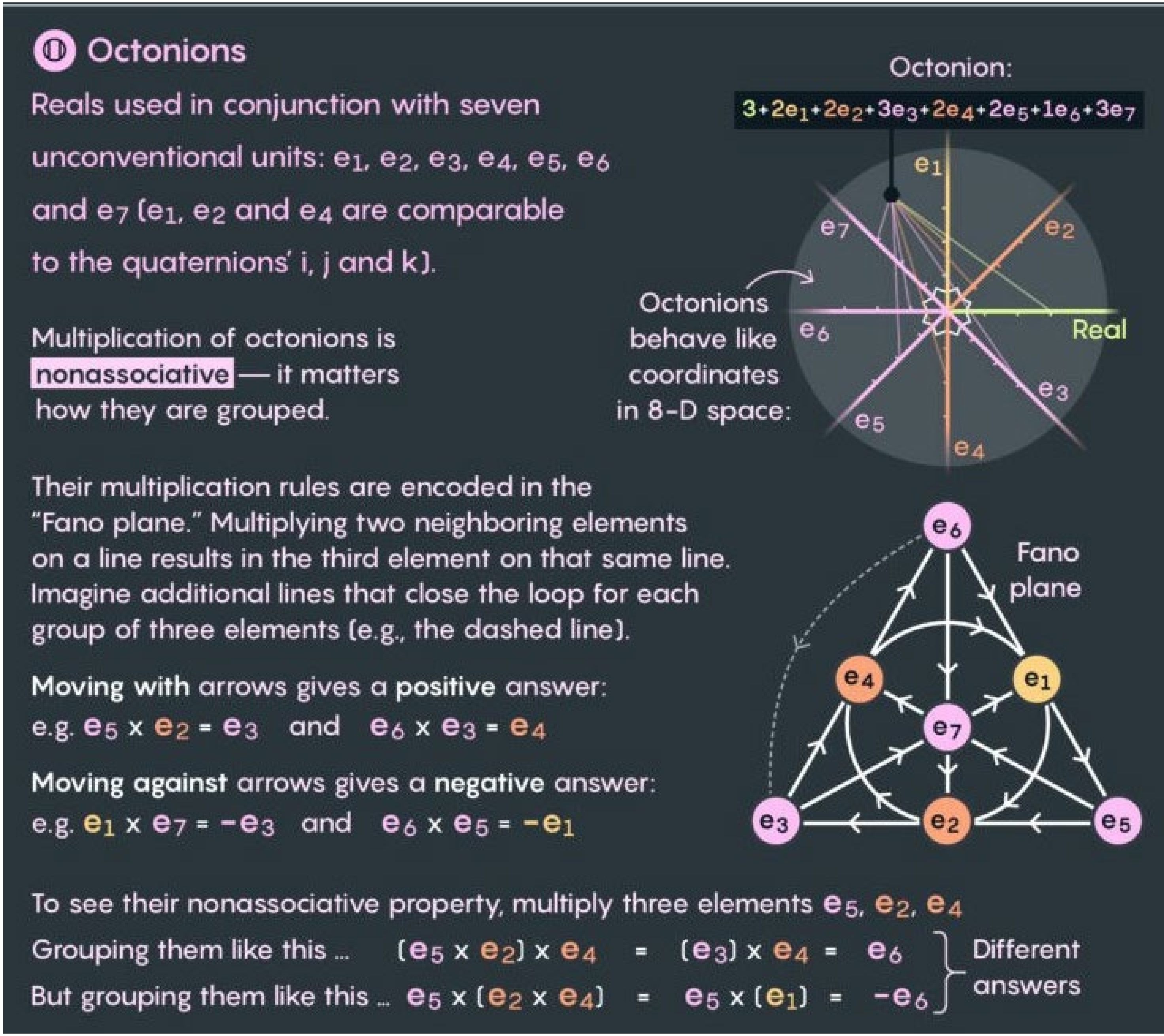
https://www.quantamagazine.org/the-octonion-math-that-could-underpin-physics-20180720/

Complicated Extensions



https://www.quantamagazine.org/the-imaginary-numbers-at-the-edge-of-reality-20181025/

https://en.wikipedia.org/wiki/Quaternion

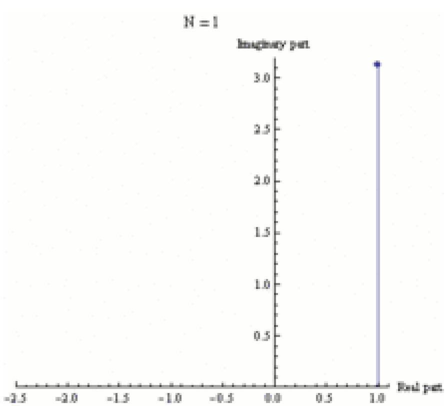


https://en.wikipedia.org/wiki/Octonion

Powers of the complex expression $(1+i\pi/n)^n$

- We can approximate numerically these complex powers without evaluating trigonometric functions

https://www.desmos.com/calculator/qtqlhrazd4

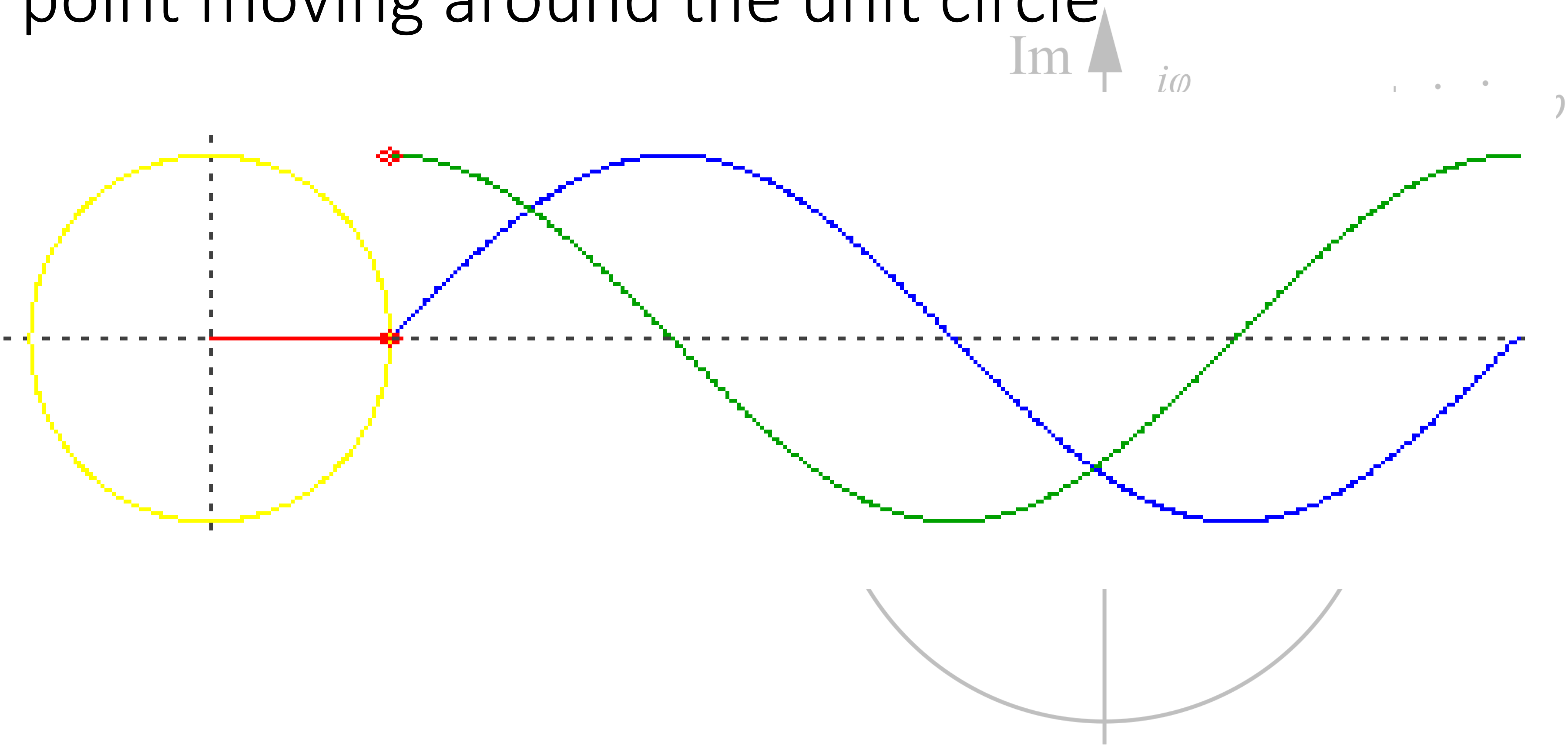


The exponential function e^z can be defined as the limit of $(1 + z/n)^n$, as n approaches infinity, and thus $e^{i\pi}$ is the limit of $(1 + i\pi/n)^n$. In this animation n takes various increasing values from 1 to 100. The computation of $(1 + i\pi/n)^n$ is displayed as the combined effect of n repeated multiplications in the complex plane, with the final point being the actual value of $(1 + i\pi/n)^n$. As n gets larger $(1 + i\pi/\mathrm{n})^n$ approaches a limit of −1.

https://www.youtube.com/watch?v=-dhHrg-KbJ0

http://demonstrations.wolfram.com/ComplexPowerPlot/

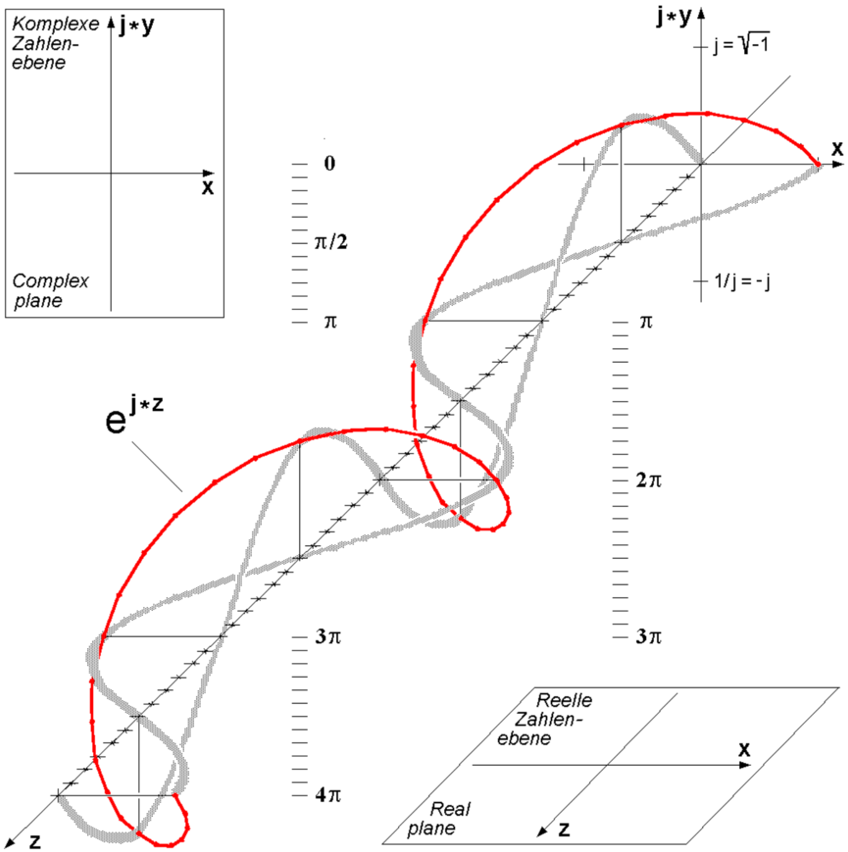
This animation depicts points moving along the graphs of the sine function (in blue) and the cosine function (in green) corresponding to a point moving around the unit circle

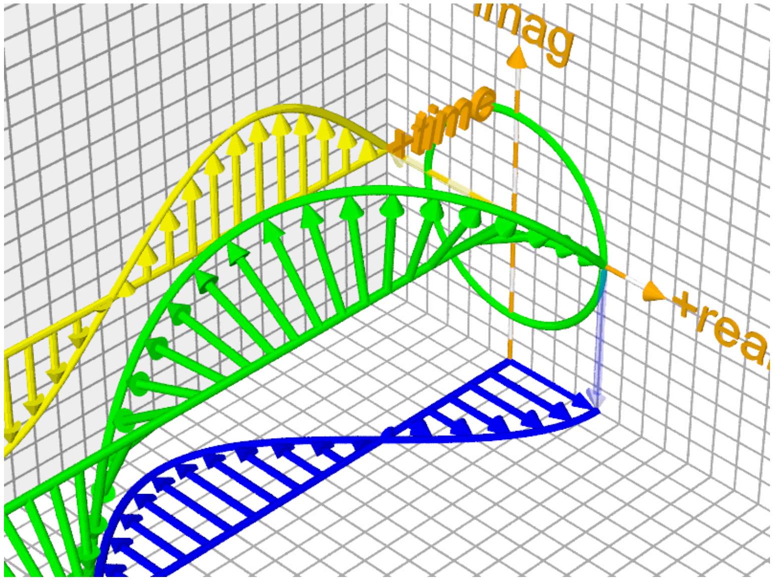


Source: http://www2.seminolestate.edu/lvosbury/AnimationsForTrigonometry.htm

$$e^{ix} = \cos x + i \sin x$$

Three-dimensional visualization of Euler's formula. See also circular polarization.





https://betterexplained.com/articles/intuitive-understanding-of-eulers-formula/

Tailor made for math

Borrowing results from calculus we can find the Taylor series expansions for the transcendental functions:

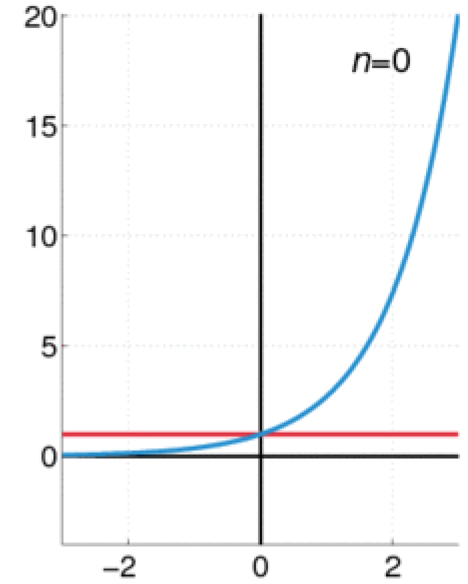
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$$

x is measured in radians

https://www.desmos.com/calculator/8ksvin4rua
https://en.wikipedia.org/wiki/Taylor_series

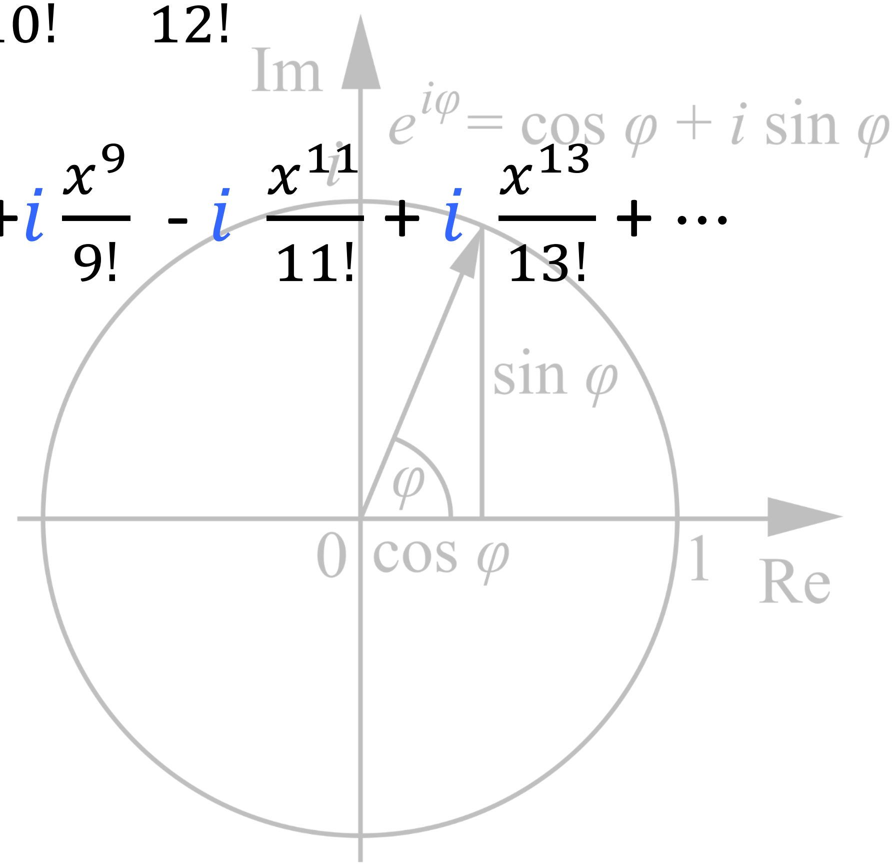


The exponential function e^x (in blue), and the sum of the first $n + 1$ terms of its Taylor series at 0 (in red).

$$e^{ix} = \cos x + i \sin x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots$$

$$i \sin x = ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + i\frac{x^9}{9!} - i\frac{x^{11}}{11!} + i\frac{x^{13}}{13!} + \cdots$$



http://www.gresham.ac.uk/lectures-and-events/eulers-exponentials

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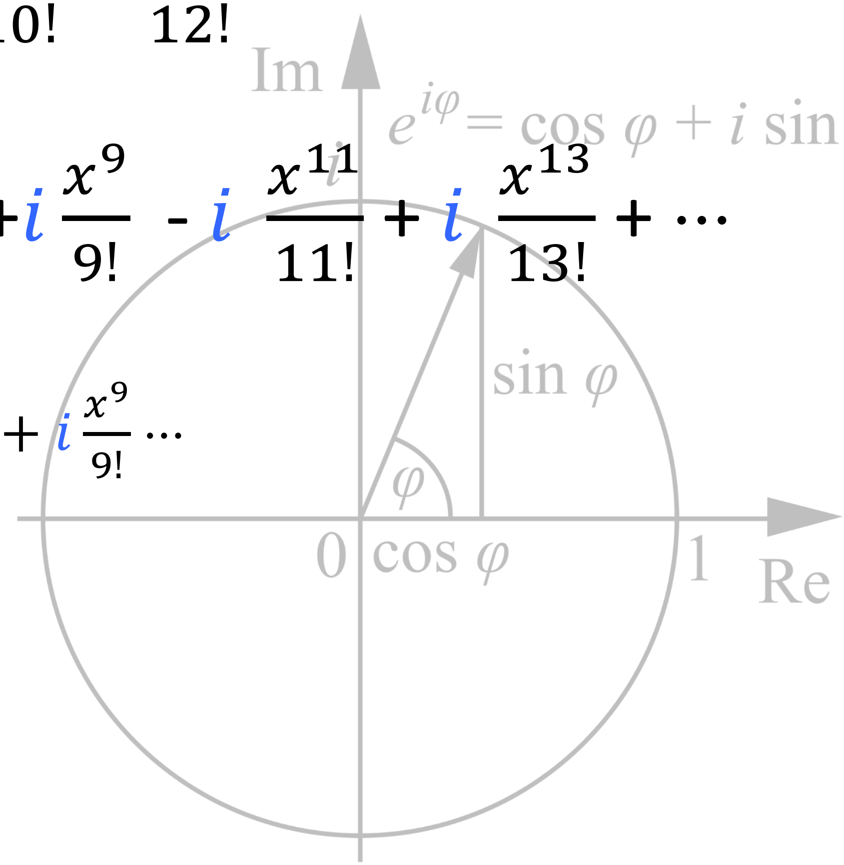
Add to get

$$1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} \cdots$$



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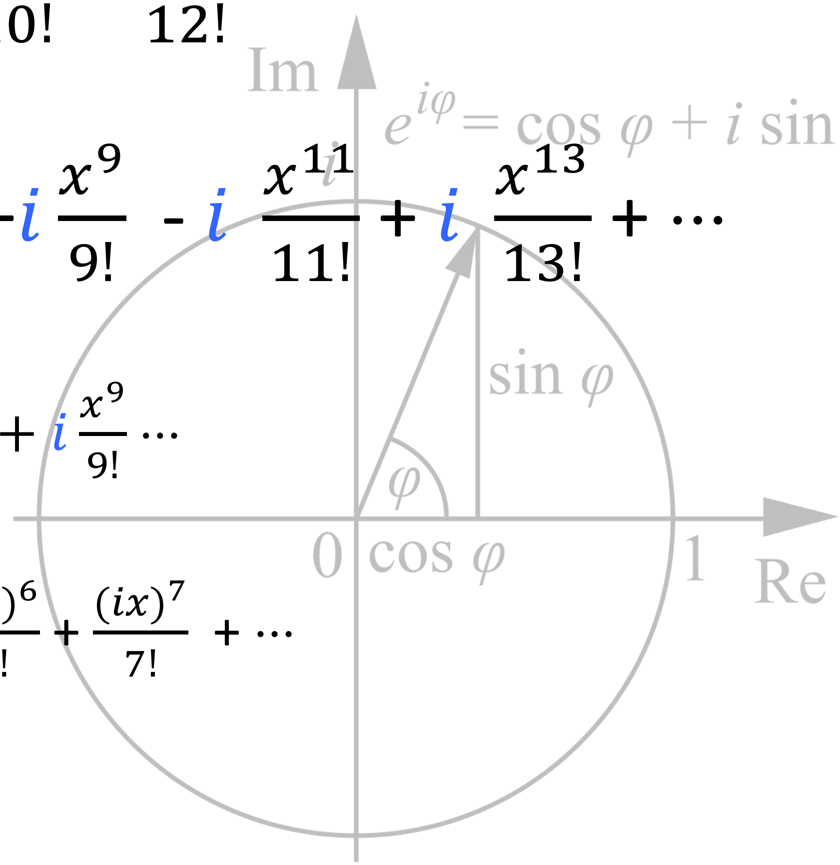
Add to get

$$1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} \cdots$$

Lets compare it with

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots$$



http://www.gresham.ac.uk/lectures-and-events/eulers-exponentials

$$e^{ix} = \cos x + i \sin x$$

Note: $i^2 = -1$

$i^3 = -i$

$i^4 = 1$

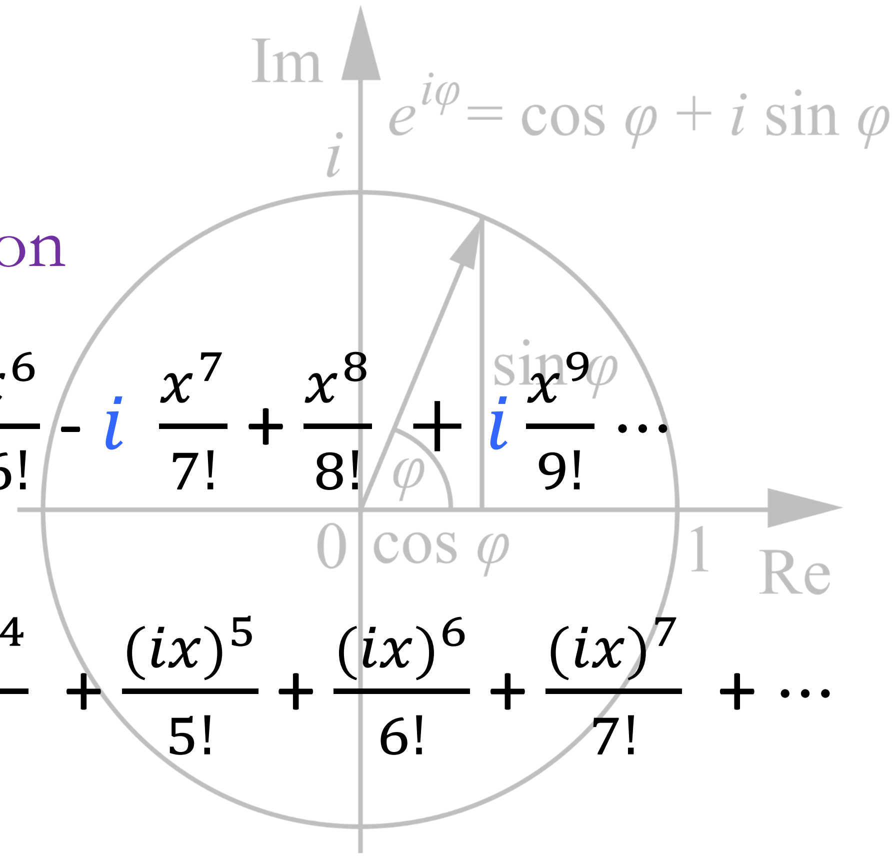
$i^5 = i$ and so on

Add to get

$$1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} \cdots$$

which is

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots$$



http://www.gresham.ac.uk/lectures-and-events/eulers-exponentials

Euler’s formula in Introductio, 1748

LIB. I.

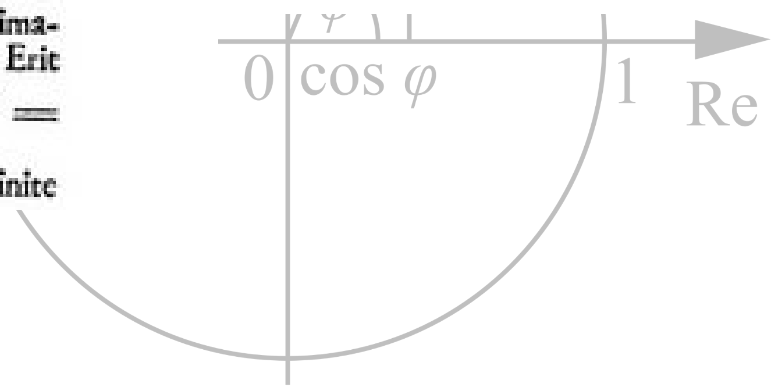
$$\frac{(1+\frac{v\sqrt{-1}}{i})^i + (1-\frac{v\sqrt{-1}}{i})^i}{2}$$; atque *fin.* $v =$

$$\frac{(1+\frac{v\sqrt{-1}}{i})^i - (1-\frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}.$$ In Capite autem

præcedente vidimus esse $(1+\frac{z}{i})^i = e^z$, denotante e basin Logarithmorum hyperbolicorum: scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit *cos.* $v =$
$\frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$ & *fin.* $v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}$.
Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Cosinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} =$ *cos.* $v + \sqrt{-1}$. *fin.* v & $e^{-v\sqrt{-1}} =$ *cos.* $v - \sqrt{-1}$. *fin.* v.

139. Sit jam in iisdem formulis §. 130. n numerus infinite

From which it can be worked out in what way the exponentials of imaginary quantities can be reduced to the sines and cosines of real arcs

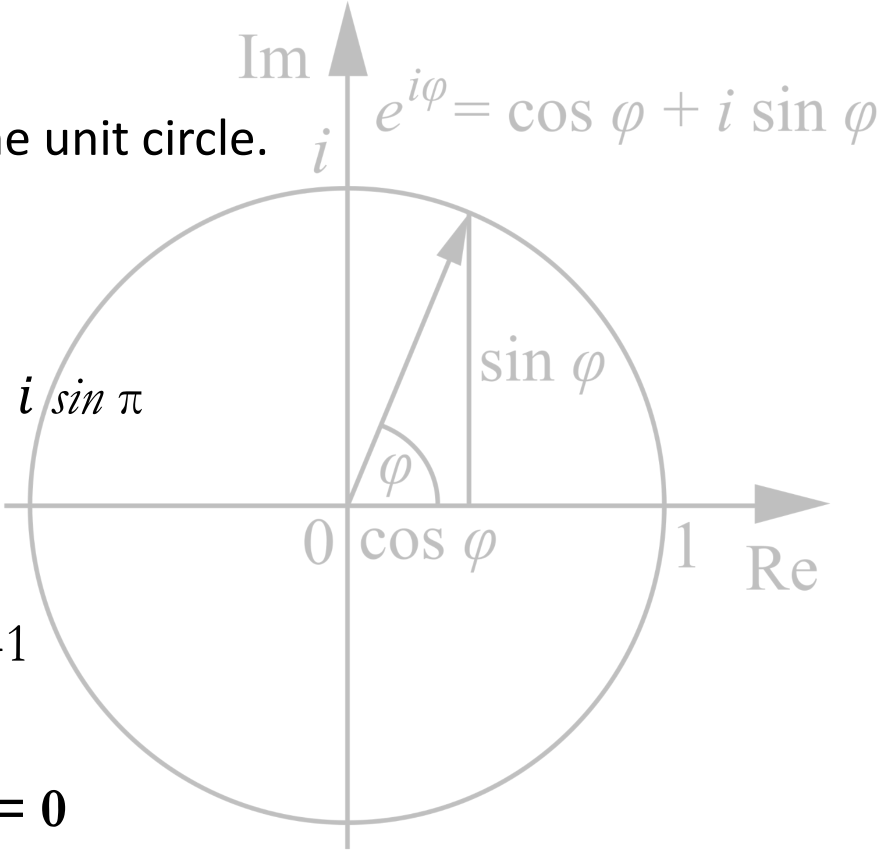


http://www.gresham.ac.uk/lectures-and-events/eulers-exponentials

$e^{ix} = \cos x + i \sin x$

Now that we have the basics, let's check how it works:

- $e^{i\left(\frac{\pi}{3}\right)} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$
- This corresponds to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle.



Now if we set x equal to π

$$e^{i\pi} = \cos\pi + i\sin\pi$$

and use $\cos\pi = -1$ and $\sin\pi = 0$ giving

$$e^{i\pi} = -1$$

or

$$\boldsymbol{e^{i\pi} + 1 = 0}$$

http://www.gresham.ac.uk/lectures-and-events/eulers-exponentials

$$e^{i\pi} + 1 = 0$$

This links five of the most important constants in mathematics:

- 0 is the additive identity, which when added to any number leaves the number unchanged
- 1 is the multiplicative identity, which multiplied by any number leaves the number unchanged
- e of the exponential function which we have defined before and base of the natural logarithms
- π which is the ratio of a circle's circumference to its diameter
- *i* is the imaginary unit, which is the square root of -1
- Richard Feynman, an incredibly famous physicist, claimed this was the jewel of mathematics. Some have written that because of the innate beauty and simplicity of this equation, that Euler used it as a proof that god must exist.

$$e^{ix} = \cos x + i \sin x$$

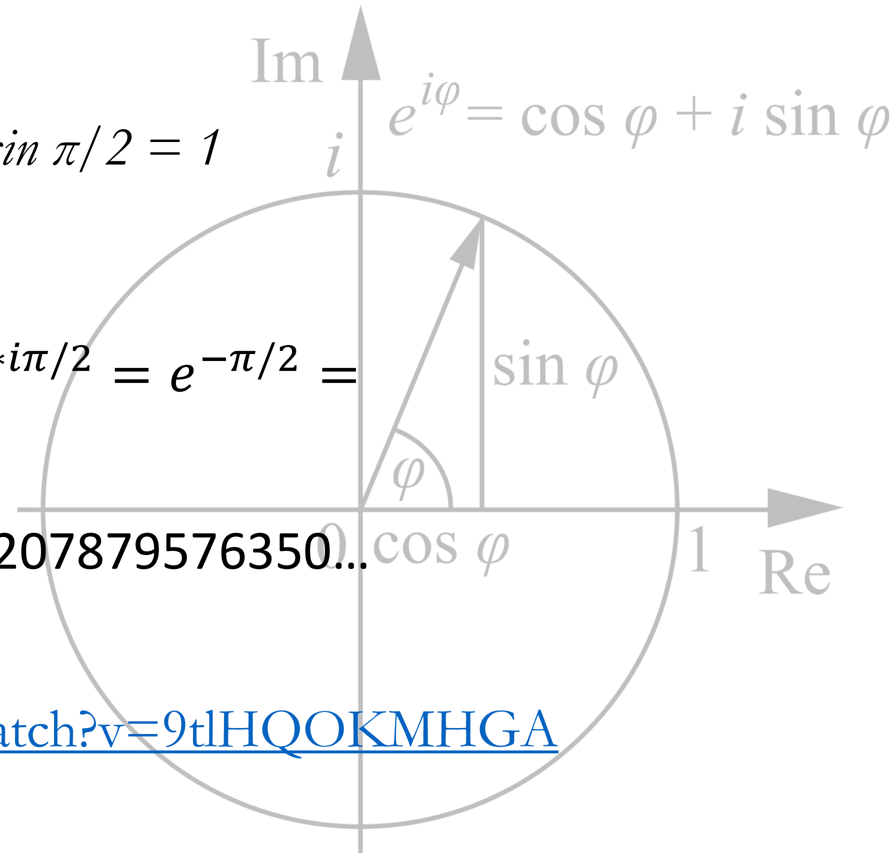
Futher unexpected results: if we raise the imaginary unit to itself as a power, the answer is a **REAL number**.

Set x equal to $\pi/2$ and use $\cos \pi/2 = 0$ and $\sin \pi/2 = 1$

Then raise both sides to the power of i.

$$i^i = (e^{i(\pi/2)})^i = e^{i*i\pi/2} = e^{-\pi/2} =$$

$$(e^{\pi})^{-1/2} = \frac{1}{\sqrt{e^{\pi}}} = 0.207879576350...$$



https://www.youtube.com/watch?v=9tlHQOKMHGA

http://www.gresham.ac.uk/lectures-and-events/eulers-exponentials

$$e^{ix} = \cos x + i \sin x$$

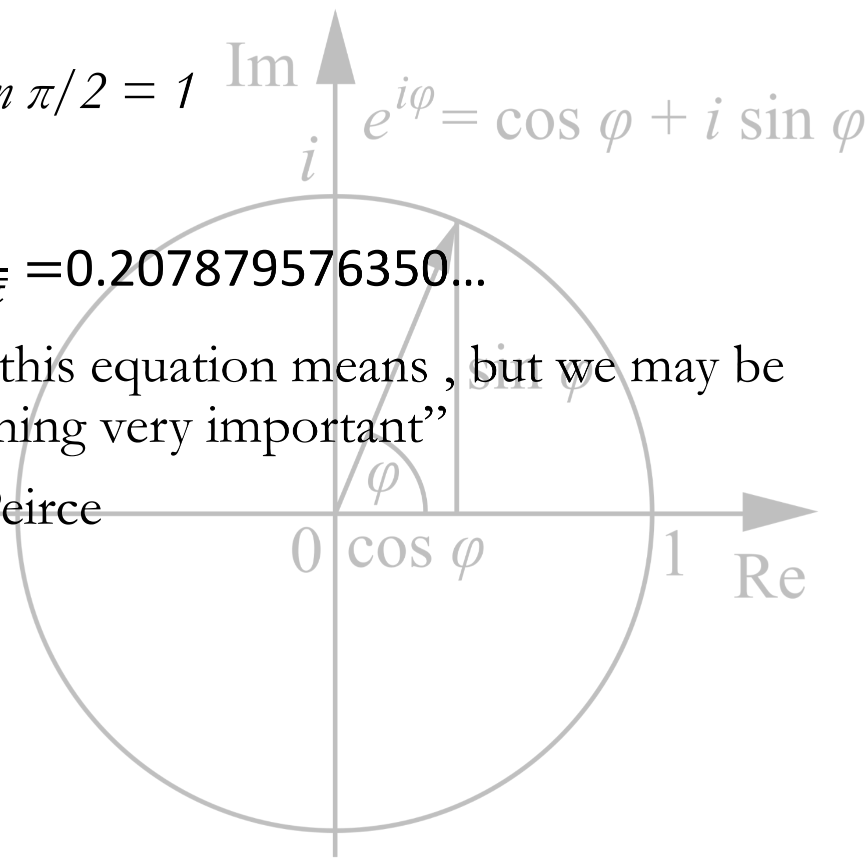
Set x equal to $\pi/2$ and use $\cos \pi/2 = 0$ and $\sin \pi/2 = 1$

Then raise both sides to the power of i.

$$i^i = \frac{1}{\sqrt{e^{\pi}}} = 0.207879576350\ldots$$

"… we have not the slightest idea of what this equation means , but we may be certain that it means something very important"

Benjamin Peirce



http://www.gresham.ac.uk/lectures-and-events/eulers-exponentials

Why is this formula so powerful?

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y) \qquad (1)$$

$$e^{i(x+y)} = e^{ix}e^{iy}$$

$$e^{i(x+y)} = \big(\cos(x) + i\sin(x)\big)\big(\cos(y) + i\sin(y)\big)$$

$$e^{i(x+y)} = \cos(x)\cos(y) + i\sin(x)\cos(y) + i\sin(y)\cos(x) + i^2\sin(x)\sin(y)$$

$$e^{i(x+y)} = \cos(x)\cos(y) + i\sin(x)\cos(y) + i\sin(y)\cos(x) - \sin(x)\sin(y)$$

$$e^{i(x+y)} = \cos(x)\cos(y) - \sin(x)\sin(y) + i\left(\sin(x)\cos(y) + \sin(y)\cos(x)\right) \qquad (2)$$

At this point, we will need to equate the real parts and the imaginary parts of the two equations.

Why is this formula so powerful?

$$e^{i(x+y)} = \cos(x+y) + i\,\sin(x+y)$$

$$e^{i(x+y)} = \cos(x)\cos(y) - \sin(x)\sin(y) + i\,(\sin(x)\cos(y) + \sin(y)\cos(x))$$

Once you do this, you can see two identities with just one quick expansion of the distributive property:

- Red (Real): $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- Green (Imaginary): $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$

You could continue doing this with $e^{i(x-y)}$ as well; no pictures needed, and very little algebra. As a bonus, the derivation is extremely quick, and you get two formulas each time!

Why is this formula so powerful?

What about other identities? Maybe Even/Odd?

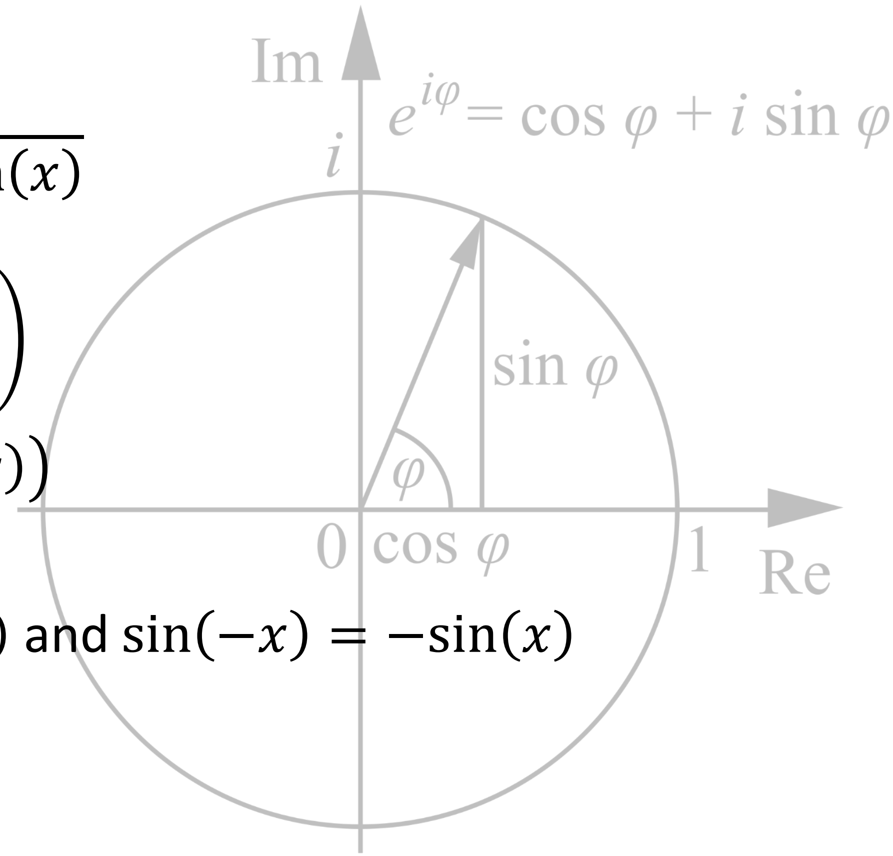
$$e^{i(-x)} = \cos(-x) + i\sin(-x)$$

$$e^{i(-x)} = e^{-ix} = \frac{1}{e^{ix}} = \frac{1}{\cos(x) + i\sin(x)}$$

Multiply by the complex conjugate.

$$\frac{1}{\cos(x) + i\sin(x)}\left(\frac{\cos(x) - i\sin(x)}{\cos(x) - i\sin(x)}\right)$$

$$= \frac{\cos(x) - i\sin(x)}{\cos^2(x) + \sin^2(x)} = \cos(x) + i(-\sin(x))$$



So this means that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$

No pictures… just a little algebra!

Application to Trigonometric Identities

What about other identities?

$$e^{i2\theta} = \cos 2\,\theta + i \sin 2\,\theta$$

On the other hand,

$$e^{i2\theta} = \left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2$$
$$= \cos^2\theta - \sin^2\theta + i(2\cos\theta\sin\theta)$$

Equating real and imaginary parts of the two expressions yield *the double angle* identities:

$$\boxed{\begin{aligned}\cos 2\,\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\,\theta &= 2\cos\theta\sin\theta\end{aligned}}$$

Application to Trigonometric Identities (cont.)

In general,

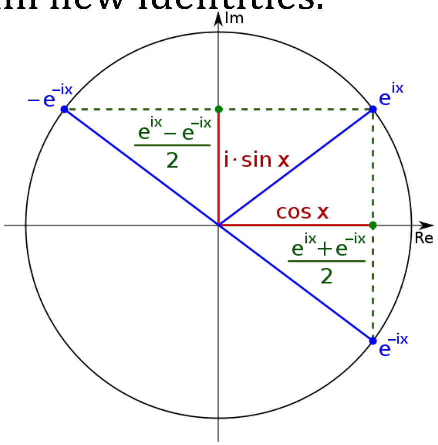
$$e^{in\theta} = \cos n\,\theta + i\sin n\,\theta = \left(e^{i\theta}\right)^n = \underbrace{(\cos\theta + i\sin\theta)^n}$$

Expand using binomial theorem, then equate real and imaginary parts to obtain new identities.

$$e^{iz} = \cos z + i\sin z$$

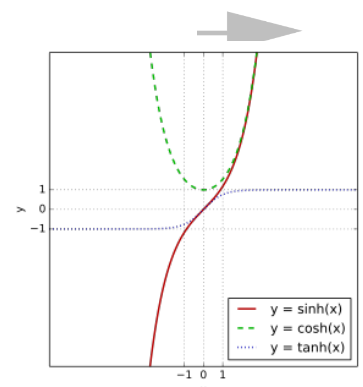
$$e^{-iz} = \cos z - i\sin z$$

$$\Rightarrow \cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$



$$\Rightarrow \cos(iz) = \frac{e^{z} + e^{-z}}{2} = \cosh z\,, \sin(iz) = -\frac{e^{z} - e^{-z}}{2i} = i\sinh z$$

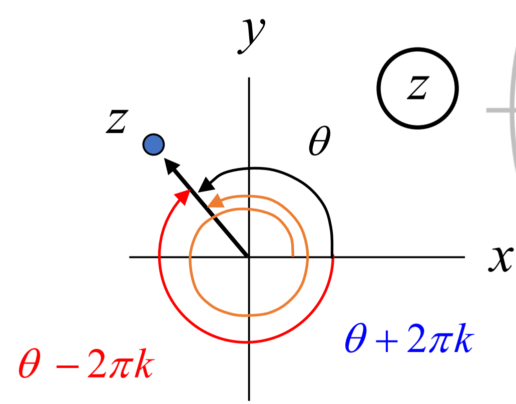
Connecting them to the hyperbolic cosine and hyperbolic sine

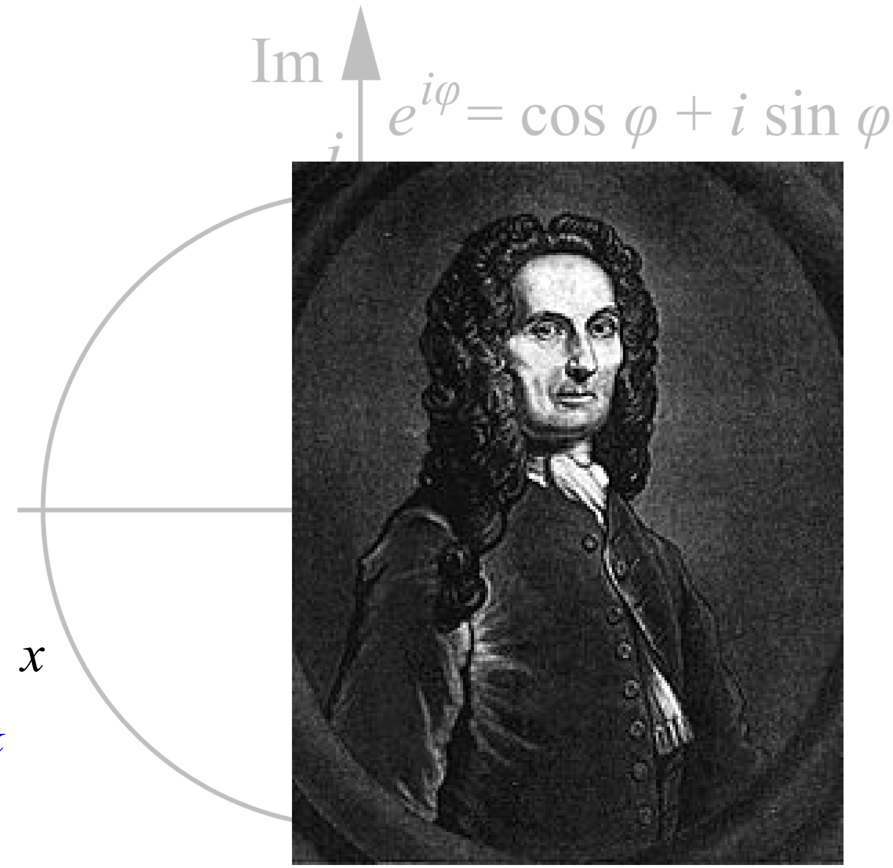


https://en.wikipedia.org/wiki/Hyperbolic_function

DeMoivre's Theorem

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n(\cos n\,\theta + i\sin n\,\theta) = r^n\angle n\theta$$





https://en.wikipedia.org/wiki/De_Moivre%27s_formula

Roots of a Complex Number

$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n(\cos n\,\theta + i\sin n\,\theta) = r^n\angle n\theta$ (DeMoivre′s Theorem)

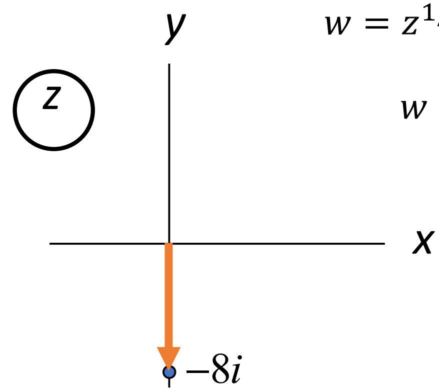
Applies also for n not an integer, but in this case, the result

Example: n th root of a complex number:

φ

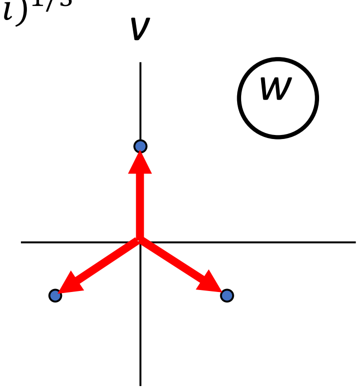
Roots of a Complex Number (cont.)

$$(-8i)^{\frac{1}{3}} = \begin{cases} \sqrt{3} - i, \\ 2i, \\ -\sqrt{3} - i \end{cases}$$



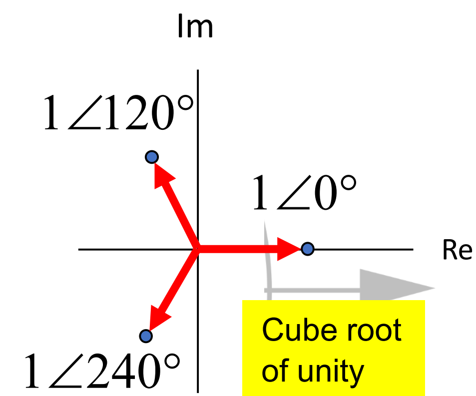
$w = z^{1/3} = (-8i)^{1/3}$

$w = u + iv$



$+ i \sin \varphi$

Note that the n th root of z can also be expressed in terms



Could we do more?

Heck yes! We could use this to find the sum to product formulas, but they require the ability to remember a substitution in the middle of the problem. These types of substitutions do come up in calculus, but we'll leave them off for now.

What kind of substitutions?

When you look at x, you probably don't immediately think $x = x + 0$.

Further, you probably don't think $x + 0 = x + \frac{y}{2} - \frac{y}{2}$, right?

Oh, and once you do think about these, do you immediately think:

$x + \frac{y}{2} - \frac{y}{2} = \frac{x}{2} + \frac{x}{2} + \frac{y}{2} - \frac{y}{2} = \frac{x}{2} + \frac{y}{2} + \frac{x}{2} - \frac{y}{2} = \frac{x+y}{2} + \frac{x-y}{2}$. ☹

Well, those are the types of substitutions needed for the sum-to-product formulas.

Why use it?

Euler's formula is a tool you can use, but don't have to. Like many things in math, it is extremely useful at times and not as useful in other situations.

Try it out. As you get comfortable with it, you may find some very interesting results!

Bonus, this opens up a whole new world. Indeed, if $e^{i\pi} = -1$, which we confirmed that it did on an earlier slide, then we could rewrite this using logarithms: $e^{i\pi} = -1 \leftrightarrow ln(-1) = i\pi$

So logarithms could be defined over *negative and even complex* numbers if we allowed complex number outputs. Test this on your calculator in complex mode to see the result. Back when covering logarithms, the domain was restricted to all non-negative real numbers… but that was needed to get the result to be a real number. Expanding our definition/domain will allow us to do MORE, not less. Now, go explore for yourselves!

https://math.stackexchange.com/questions/2089690/log-of-a-negative-number

https://en.wikipedia.org/wiki/Complex_logarithm

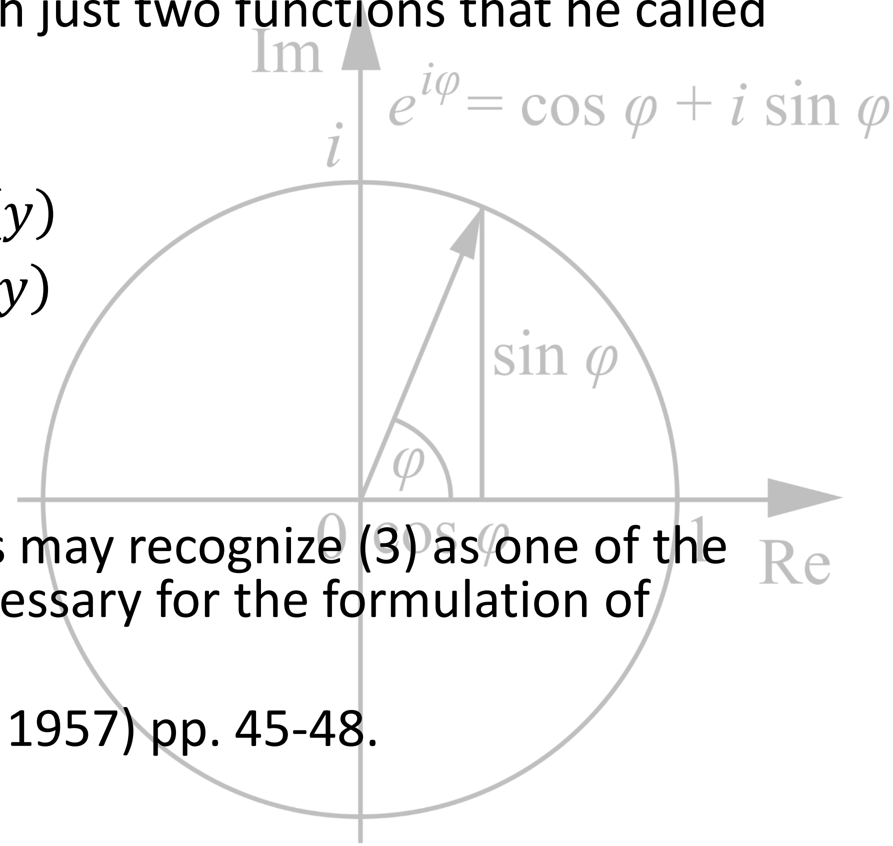
How Important are the Addition Formulas?

Back in the 1950s, Professor Hans Rademacher* showed that all of trigonometry could be developed with just two functions that he called "C" and "S" where:

1. $C(x-y) = C(x)C(y) + S(x)S(y)$
2. $S(x-y) = S(x)C(y) - S(x)C(y)$
3. $\lim_{x \to 0^+} \frac{S(x)}{x} = 1$

Those of you who have taken calculus may recognize (3) as one of the most important limits in calculus, necessary for the formulation of trigonometric derivatives.

*Mathematics Teacher Vol L (January 1957) pp. 45-48.



Calc link (power reduction)

$$\int cos^2(x)dx = \int \frac{cos(2x) + 1}{2} dx$$

φ

$$e^{ix} = \cos(x) + i\sin(x)$$
$$e^{-ix} = \cos(x) - i\sin(x)$$

So $e^{ix} + e^{-ix} = 2\cos(x)$ which means $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$

This means $cos^2(x) = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 = \frac{e^{i2x}+2+e^{-i2x}}{4} = \frac{2cos(2x)+2}{4} = \frac{cos(2x)+1}{2}$

Calc link (power reduction)

Expanding the technique from the previous page, we can even do something like this:

$$\int cos^6(x)dx = \int \frac{2cos(6x) + 12cos(4x) + 30cos(2x) + 20}{64}dx$$

It helps to remember Pascal's triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Calc link (power reduction)

$\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$, which means $e^{ix} + e^{-ix} = 2\cos(x)$.

$$\cos^6(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^6 = \frac{\left(e^{ix} + e^{-ix}\right)^6}{2^6}$$

Using Pascal's Triangle for coefficients:
$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Expanding the binomial:

$$\left(e^{ix} + e^{-ix}\right)^6 = \left(e^{ix}\right)^6 + 6\left(e^{ix}\right)^5\left(e^{-ix}\right) + 15\left(e^{ix}\right)^4\left(e^{-ix}\right)^2 + 20\left(e^{ix}\right)^3\left(e^{-ix}\right)^3 + 15\left(e^{ix}\right)^2\left(e^{-ix}\right)^4 + 6\left(e^{ix}\right)\left(e^{-ix}\right)^5 + \left(e^{-ix}\right)^6$$

Simplifying exponents:

$$\left(e^{ix} + e^{-ix}\right)^6 = e^{i6x} + 6e^{i5x}\left(e^{-ix}\right) + 15e^{i4x}\left(e^{-i2x}\right) + 20e^{i3x}\left(e^{-i3x}\right) + 15e^{i2x}\left(e^{-i4x}\right) + 6e^{ix}\left(e^{-i5x}\right) + e^{-i6x}$$

Calc link (power reduction)

Simplifying exponents:

$$\left(e^{ix}+e^{-ix}\right)^6 = e^{i6x}+e^{-i6x}+6e^{i4x}+6e^{-i4x}+15e^{i2x}+15e^{-i2x}+20e^{i0x}$$

Grouping like objects:

$$\left(e^{ix}+e^{-ix}\right)^6 = \left(e^{i6x}+e^{-i6x}\right)+6\left(e^{i4x}+e^{-i4x}\right)+15\left(e^{i2x}+15e^{-i2x}\right)+20$$

Euler's formula then substitutes since $e^{ix}+e^{-ix}=2\cos(x)$:

$$\left(e^{ix}+e^{-ix}\right)^6 = 2cos6x+6(2cos4x)+15(2cos2x)+20$$

This shows why $cos^6(x)=\frac{2cos(6x)+12cos(4x)+30cos(2x)+20}{64}$.

And now you can do the integration in calculus II much quicker:

$$\int cos^6(x)dx = \int \frac{2cos(6x)+12cos(4x)+30cos(2x)+20}{64}dx$$

Lets raise the roof!

Recall:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

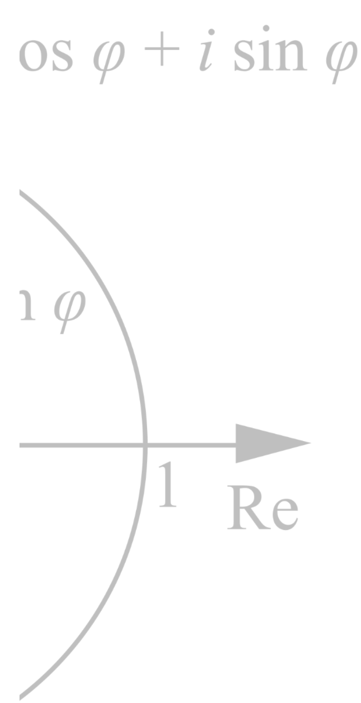
Define extension to complex variable($x \to z = x + iy$):

$$e^z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{(converges for all } z\text{)}$$

$$\Rightarrow e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\Rightarrow \boxed{e^{i\theta} = \cos\theta + i\sin\theta} \boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$$



http://mathworld.wolfram.com/ComplexExponentiation.html

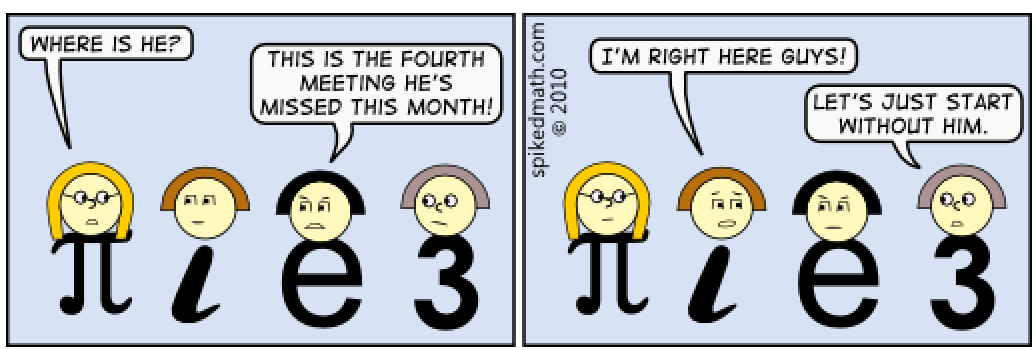
https://brilliant.org/wiki/complex-exponentiation/

Further connections

In Calculus 2 (Math 141) you will see Taylor and McLaurin series. Used to confirm Euler's formula $e^{ix} = \cos(x) + i\sin(x)$.

In Discrete Math (Math 245) you can prove that the previous formulas are true for all values of *n*, using formal mathematical induction.

Calc I and II (Math 140 and 141) often must substitute in power reduction formulas for powers of trigonometric functions.



http://spikedmath.com/297.html