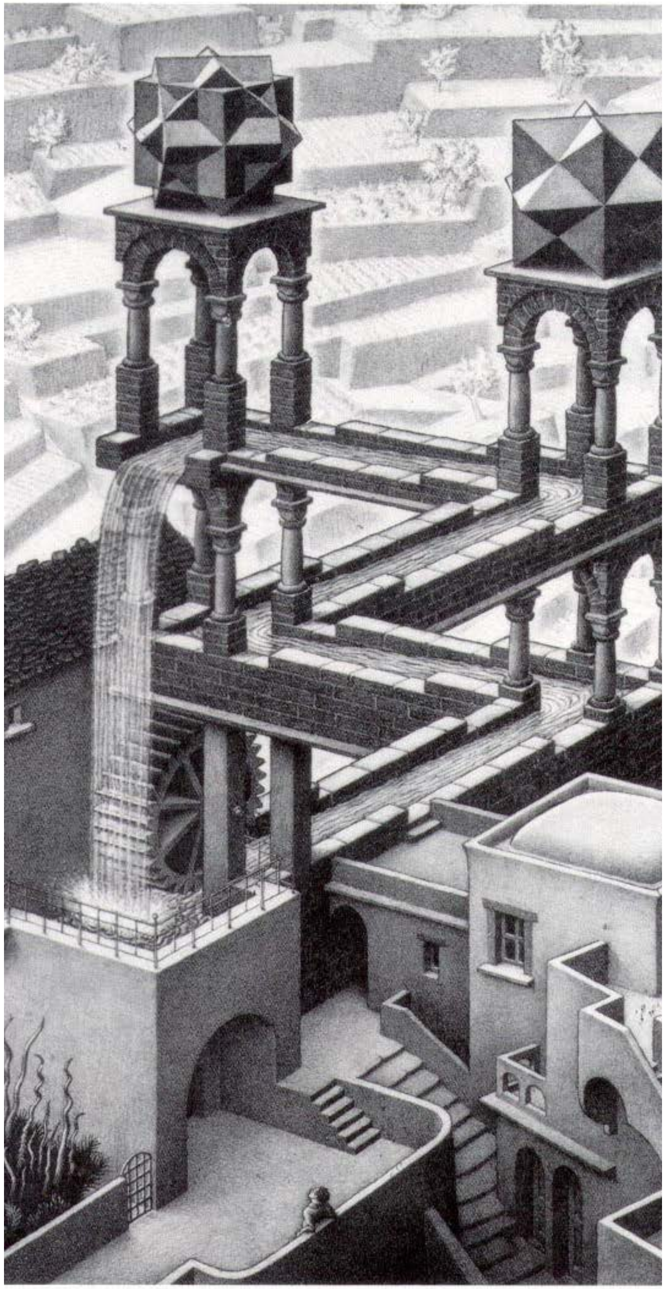
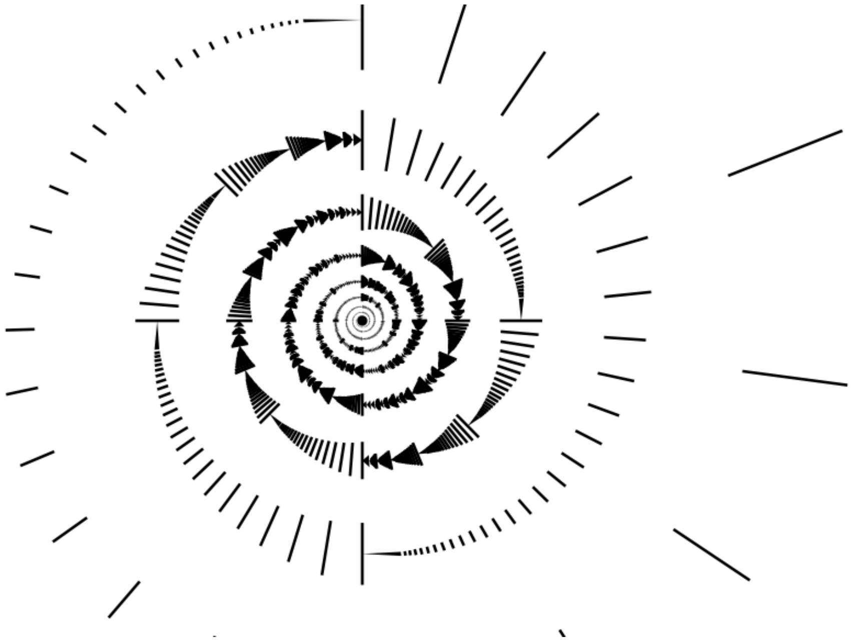




Infinity and its cardinalities

No one shall expel us from the Paradise
that Cantor has created. David Hilbert

How to use the concept of infinity coherently

- The whole difficulty of the subject lies in the necessity of thinking in an unfamiliar way, and in realizing that many properties which we have thought inherent in number are in fact peculiar to finite numbers. If this is remembered, the positive theory of infinity...will not be found so difficult as it is to those who cling obstinately to the prejudices instilled by the arithmetic which is learnt in childhood. Bertrand Russell (Salmon 1970, 58)

19th century mathematics

- In the nineteenth century mathematics experienced a movement towards a progressively abstract style with an increased emphasis on putting itself on a sound and rigorous basis and on examining its foundations. This movement happened in the calculus, algebra and geometry.
- In the 1820s the French mathematician Cauchy, the most prolific mathematician of the century, made a major advance in making the calculus rigorous by clarifying the concept of a limit. This idea of a limit is needed in the calculus, for example, where we have the ratio of two quantities and we want to see what happens to this ratio as both quantities move simultaneously towards zero so becoming infinitely small.

19th century mathematics

- It was later in the century that the German mathematician, Weierstrass, gave a mathematically and logical solid definition of a limit and it is his definition of a limit that we use today and on which the calculus is founded.

- However, as often happens, the resolution of one problem drew attention to another problem. It turned out that getting a sound definition of a limit necessitated a rigorous definition of the real numbers which in turn led to the study of infinite sets by Cantor.

Basic Set Theory

Set: *any collection into a whole M of definite and separate objects m of our intuition or of our thought*

Broadly speaking a *set* is a collection of objects.

Examples:

A={1, 3, 4, 6, 8}

B={1, 2, 3, …, 66}

C={2, 4, 6, 8, …}

D={n ∈ $\mathbb{Z}$ | n is even}

E={x ∈ $\mathbb{N}$ | x is a prime number $\leq 10^6$}

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.
- We write *A* = *B* if *A* and *B* are equal sets.
- For example, in set notation order and repetition does not matter since:

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Cardinality

The number of elements in a set is called the **cardinal number**, or **cardinality** of the set.

The symbol ***n*(*A*)**, read "***n*** of ***A***," represents the cardinal number of set *A*.

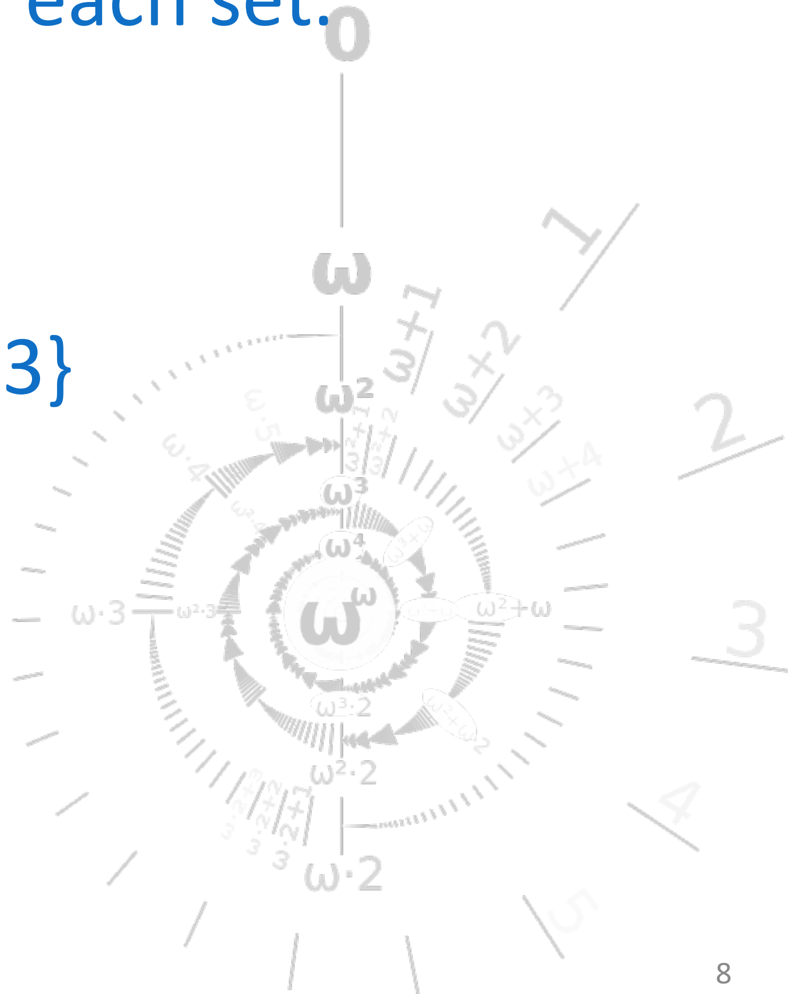
Cardinality

Find the cardinal number of each set.

a) $K = \{a, l, g, e, b, r\}$

b) $M = \{0\}$

c) $C = \{\,13, 14, 15, \ldots, 22, 23\}$



Solution

a) $n(K) = 6$

b) $n(M) = 1$

c) $n(C) = 11$

Finite and Infinite Sets

If the cardinal number of a set is a particular whole number, we call that set a **finite set**.

Whenever a set is so large that its cardinal number is not found among the whole numbers, we call that set an **infinite set**. This is Intuitive but an informal definition.

Galileo's Paradox of Equinumerosity

"There are as many squares as there are numbers because they are just as numerous as their roots." — Galileo Galilei, 1632

Consider the set of natural numbers N = {1, 2, 3, 4, …} and the set of perfect squares (i.e. the squares of the naturals) S = {1, 4, 9, 16, 25, …}. After careful thought, Galileo produced the following contradictory statements regarding these two sets:

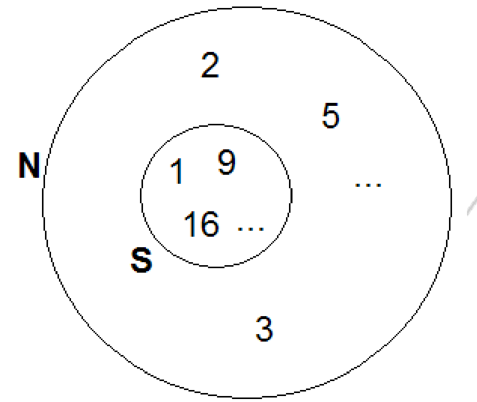
1 – While some natural numbers are perfect squares, some are clearly not. Hence the set N must be more numerous than the set S, or $|N|>|S|$.

2 – Since for every perfect square there is exactly one natural that is its square root, and for every natural there is exactly one perfect square, it follows that S and N are equinumerous, or $|N|=|S|$.

Galileo's Paradox of Equinumerosity

Here, Galileo's exact matching of the naturals with the perfect squares constitutes an early use of a **one-to-one correspondence** between sets – the conceptual basis for Cantor's theory of sets.

N	1	2	3	4	5	…	n	…
	↕	↕	↕	↕	↕		↕	
S	1	4	9	16	25	…	n^2	…



To resolve the paradox, Galileo concluded that the concepts of "less," "equal," and "greater" were inapplicable to the cardinalities of infinite sets such as S and N, and could only be applied to finite sets.

In fact, Cantor would prove that, in general, this is not true. He showed that **some infinite sets have a greater cardinality than others**, thus implying the existence of different "sizes" for infinity.

Equivalent Sets

Set *A* is **equivalent** to set *B* if and only if $n(A) = n(B)$.

Example:

D={ a, b, c }; E={apple, orange, pear}

$n(D) = n(E) = 3$

So set *A* is equivalent to set *B*.

Equivalent Sets

- Any sets that are equal must also be equivalent.
- Not all sets that are equivalent are equal.

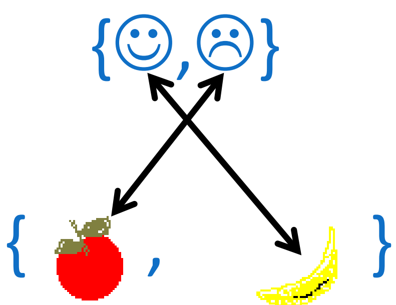
Example: D ={ a, b, c }; E ={apple, orange, pear}

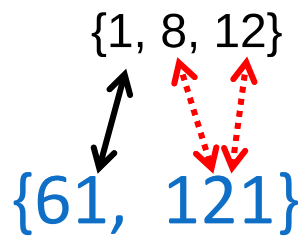
n(D) = n(E) = 3; so set A is equivalent to set B, but the sets are NOT equal, since they do not contain the elements of each other.

Cardinality and Finite Sets

DEF: Two sets *A* and *B* have are in one to one correspondence (bijection) if there's a pairing (function) between the elements of one set to the other such that:

for each element of one set there is one and only one partner on the other set (function is one-to-one), and vice versa. There are no unpaired elements (function is onto).





Infinite Set

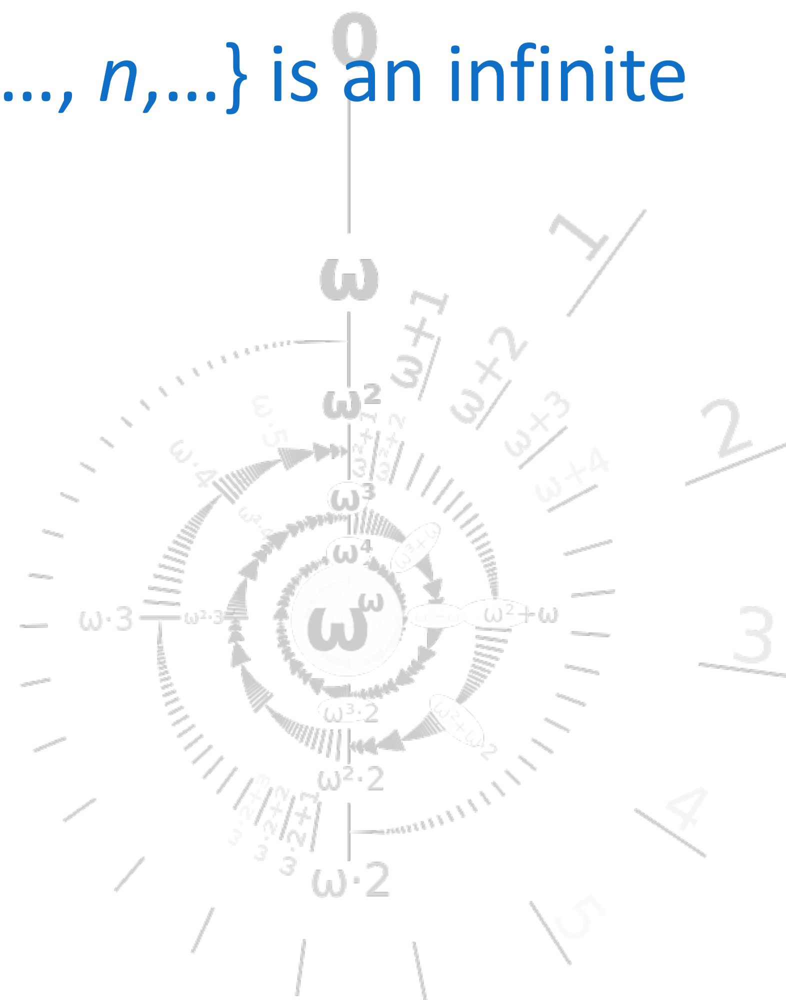
- An **infinite set** is a set that can be placed in a one-to-one correspondence with a proper subset of itself.

- A **proper** subset does not contain all the elements of the set.

- This a nonintuitive definition, that is more formal, and independent of the notion of cardinality. It takes us away from our “finite” experience.

The Set of Natural Numbers

Show that $N = \{1, 2, 3, 4, 5, \ldots, n, \ldots\}$ is an infinite set.



The Set of Natural Numbers

- Solution

Remove the first element of set *N*, to get the proper subset *P* of the set of counting numbers

$N = \{1, 2, 3, 4, 5, \ldots, \quad n, \ldots\}$

$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow$

$P = \{2, 3, 4, 5, 6, \ldots, n + 1, \ldots\}$

For any number *n* in *N*, its corresponding number in *P* is $n + 1$.

An explicit one-to-one and onto function (bijection) $f:N \rightarrow P$ is given by $f(n)=n+1$

The Set of Natural Numbers

- Solution

We have shown the desired one-to-one correspondence, therefore the set of counting numbers is infinite.

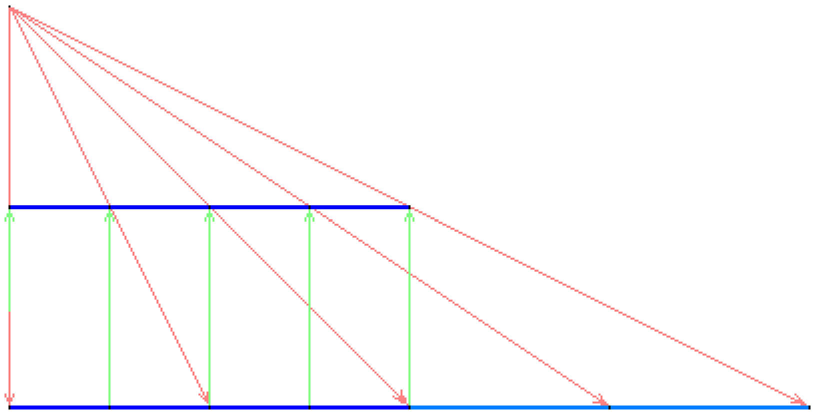
$N = \{1, 2, 3, 4, 5, \ldots, \quad n, \ldots\}$

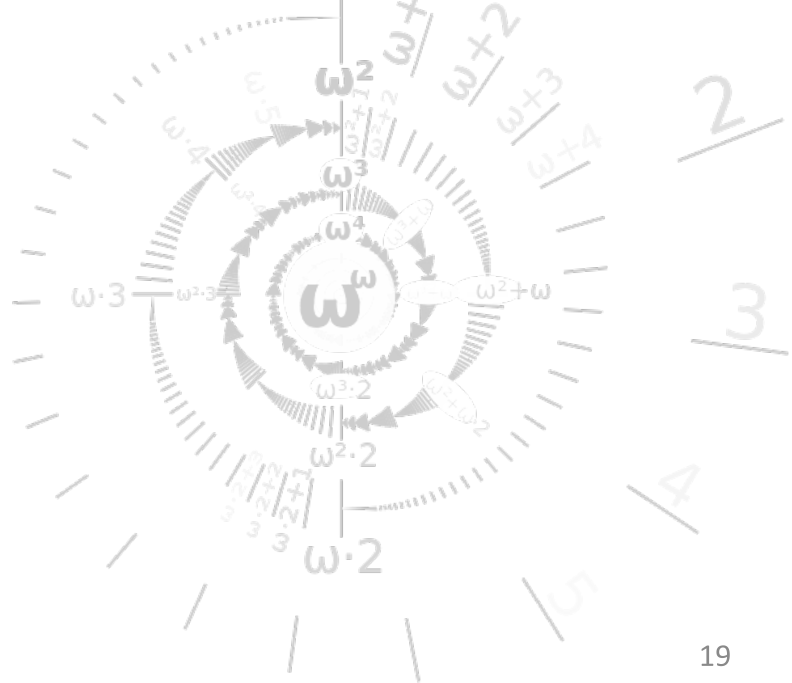
$\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \qquad \updownarrow$

$P = \{2, 3, 4, 5, 6, \ldots, n + 1, \ldots\}$

Understanding Proper subsets

- Careful visualizing One-to-one correspondence between an infinite set and its proper subsets
- Not including some of the elements of the original set does not disqualify having the same cardinality





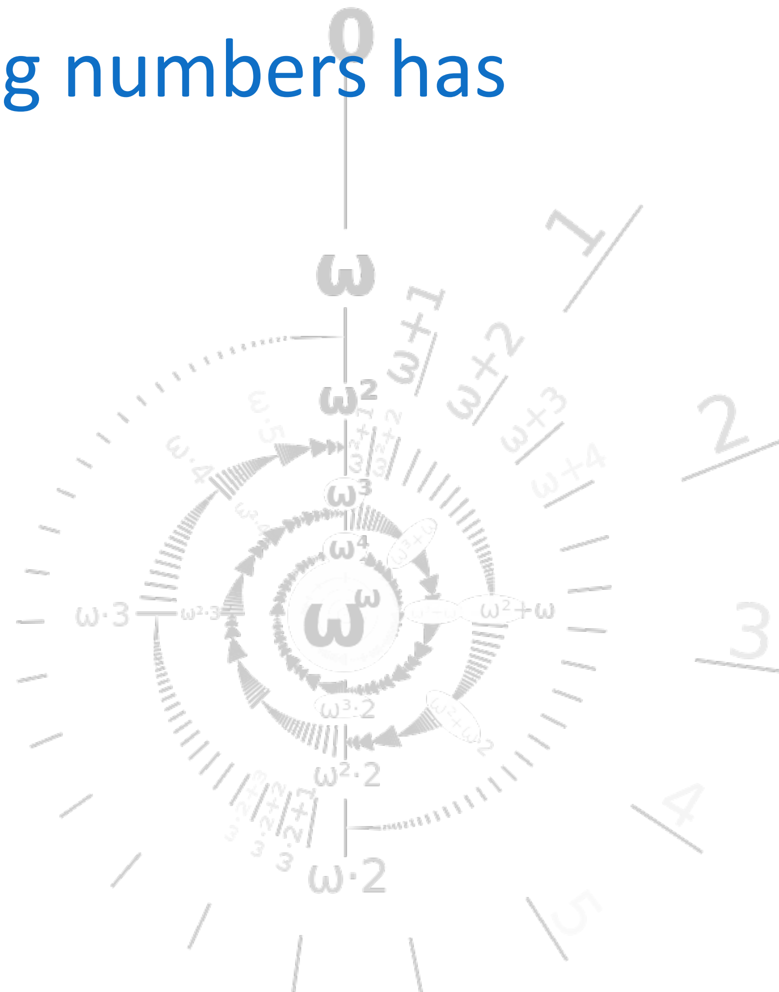
Countable Sets

- A set is countable if it is finite or if it can be placed in a one-to-one correspondence with the set of counting numbers (countable infinite).

- All infinite sets that can be placed in a one-to-one correspondence with a set of counting numbers have cardinal number aleph-naught or aleph-zero, symbolized $\aleph_0$.

The Cardinal Number of the Set of Odd Numbers

Show the set of odd counting numbers has cardinality $\aleph_0$.



The Cardinal Number of the Set of Odd Numbers

- Solution

We need to show a one-to-one correspondence between the set of counting numbers and the set of odd counting numbers.

$$N = \{1, 2, 3, 4, 5, \ldots, \quad n, \ldots\}$$
$$\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \qquad \updownarrow$$
$$O = \{1, 3, 5, 7, 9, \ldots, 2n-1, \ldots\}$$

An explicit one-to-one and onto function (bijection) f:N→O is given by f(n)=2n-1

The Cardinal Number of the Set of Odd Numbers

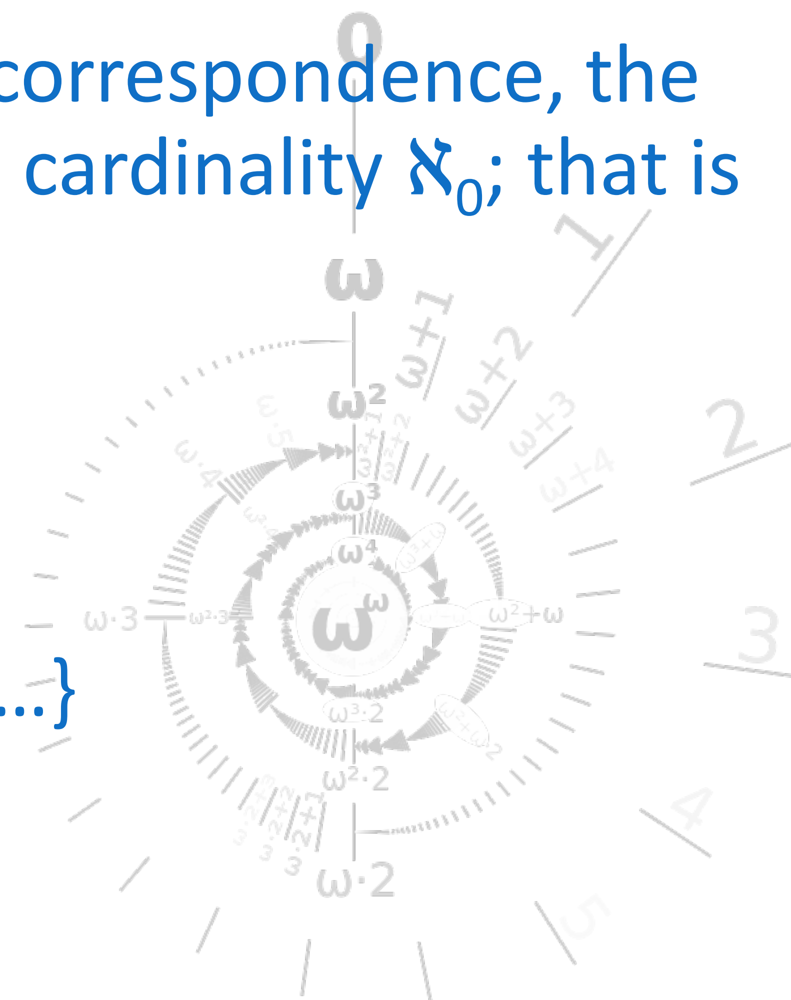
- Solution

Since there is a one-to-one correspondence, the odd counting numbers have cardinality $\aleph_0$; that is $n(O) = \aleph_0$.

$$N = \{1, 2, 3, 4, 5, \ldots, \quad n, \ldots\}$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow$$

$$O = \{1, 3, 5, 7, 9, \ldots, 2n-1, \ldots\}$$



Infinite Countable Sets

- By similar one to one correspondences, one can show that the following sets have cardinality $\aleph_0$.

- Even numbers
- Perfect squares, cubes or perfect nth powers
- Prime numbers
- Fibonacci numbers
- Any infinite subset of the counting numbers

The Cardinal Number of the Integers

- That makes sense, $\aleph_0$ is the smallest infinity.
- *Obviously* we need bigger sets to possibly find bigger sizes of infinity
- Integers = {…-3,-2,-1,0,1, 2,3,…}
- Is the set of all integers countable infinite?
- Can we find a pairing (1-to-1 correspondence) with the counting numbers?

The Cardinal Number of the Integers

Since there is a one-to-one correspondence, the Integers have cardinality $\aleph_0$

$$N = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7, \ldots,$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$Z = \{0, 1,\ -1,\ 2,\ -2, 3,\ -3 \ldots\}$$

An explicit one-to-one and onto function (bijection) f:N→Z is given by

$$f(n) = \begin{cases} \frac{n}{2}\ if\ n\ is\ even \\ -\left(\frac{n-1}{2}\right)\ if\ n\ is\ odd \end{cases}$$

An explicit one-to-one and onto function (bijection) f:Z→N is given by

$$f(n) = \begin{cases} 2n\ if\ n\ is\ positive \\ 2|n| + 1\ if\ n\ is\ nonpositive \end{cases}$$

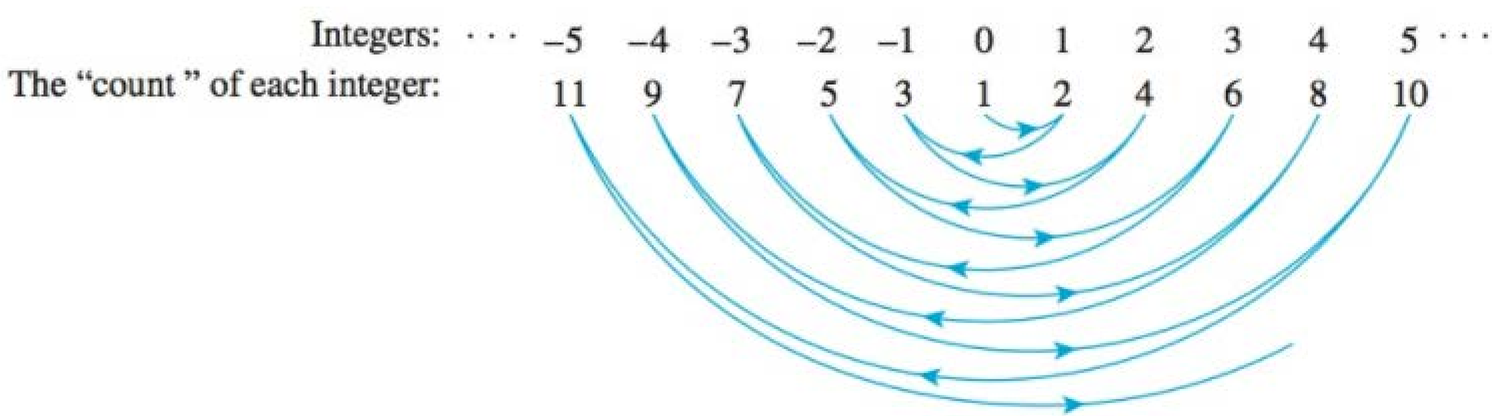
The Cardinal Number of the Integers

Since there is a one-to-one correspondence, the Integers have cardinality $\aleph_0$

$$N = \{1, 2, 3, 4, 5, 6, 7, \ldots,$$

$$\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow$$

$$Z = \{0, 1, -1, 2, -2, 3, -3 \ldots\}$$



"Counting" the Set of All Integers

"cut number line in half" and interleave the integers

Hilbert's Paradox of the Grand Hotel

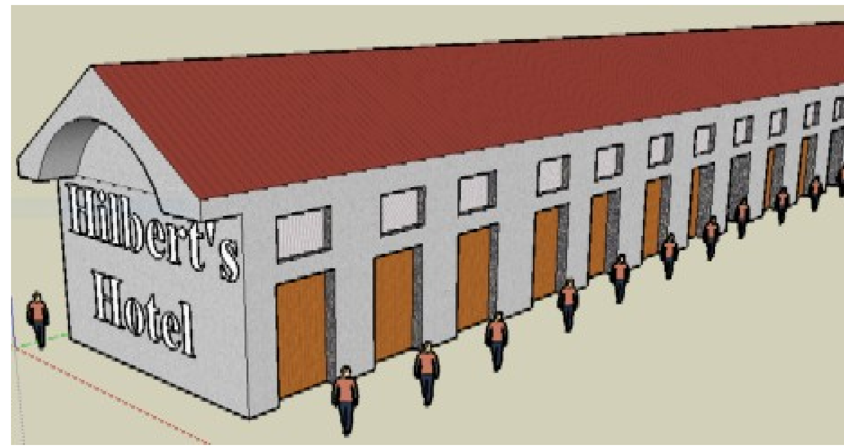
- In the early 1900's, the German mathematician David Hilbert proposed a hypothetical scenario that keenly illustrates Cantor's counter-intuitive results on infinite sets.

- He called it the paradox of the Grand Hotel (or Hotel Infinity).

- Consider a hotel with infinitely many rooms, all of which are occupied (filled at capacity).

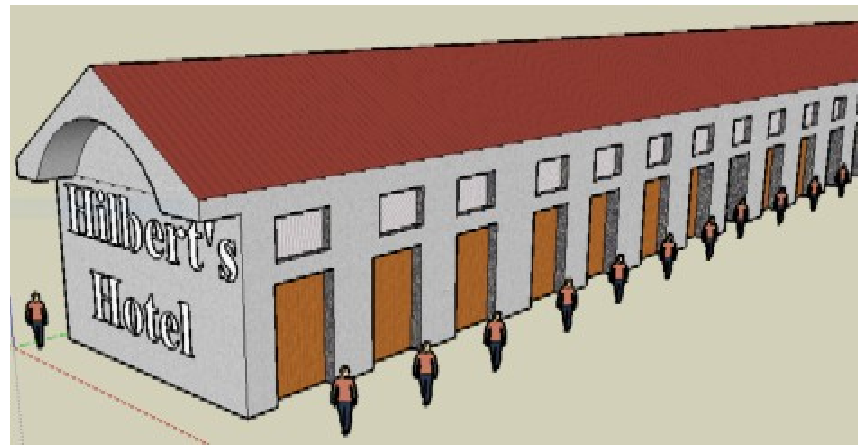
- He showed that it was possible then to accommodate a **countably infinite** number of passengers arriving at the hotel in a **countably infinite** number of buses!

Hilbert's Grand Hotel



- Lets start analysing finite new arrivals to the filled hotel.
- One new arrival

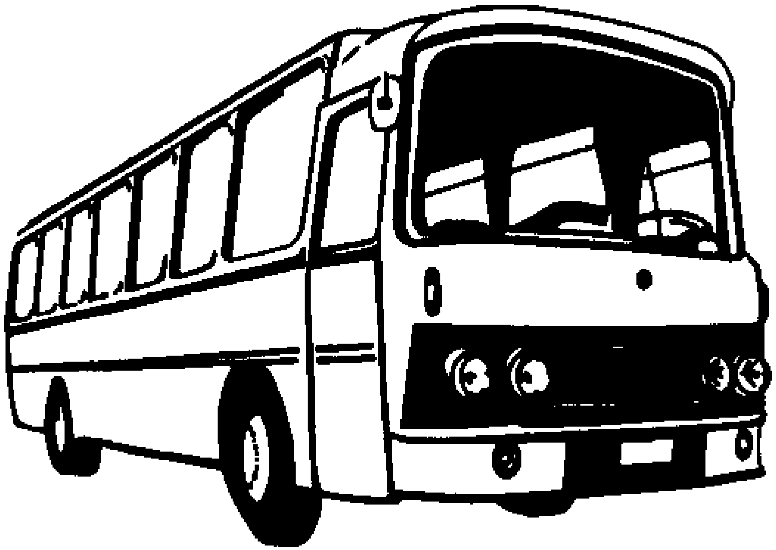
Hilbert’s Grand Hotel

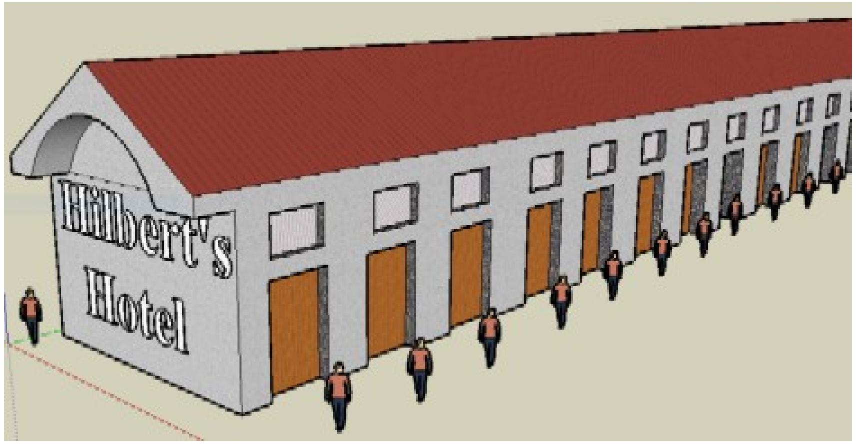


- One new arrival
- Everybody moves up ONE room.
- Since no LAST room! Everyone gets a room.
- New arrival put in room 1
- Done!
- Correspondence n → n+1
- $1 + \aleph_0 = \aleph_0$

Hilbert’s Grand Hotel

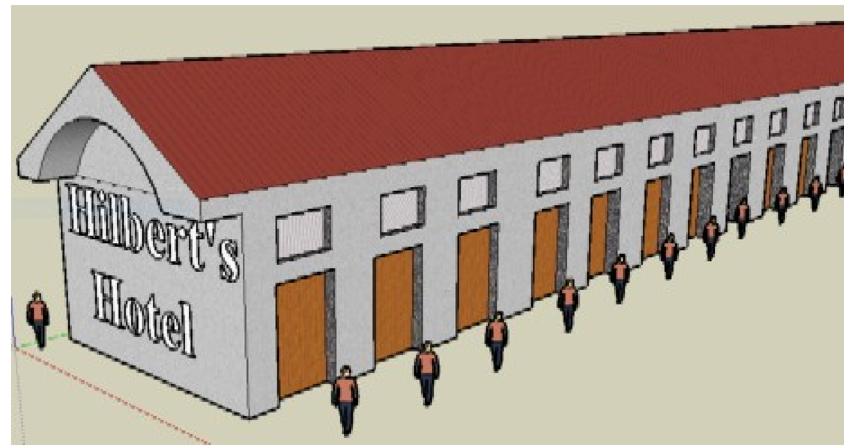
- 259 new arrivals

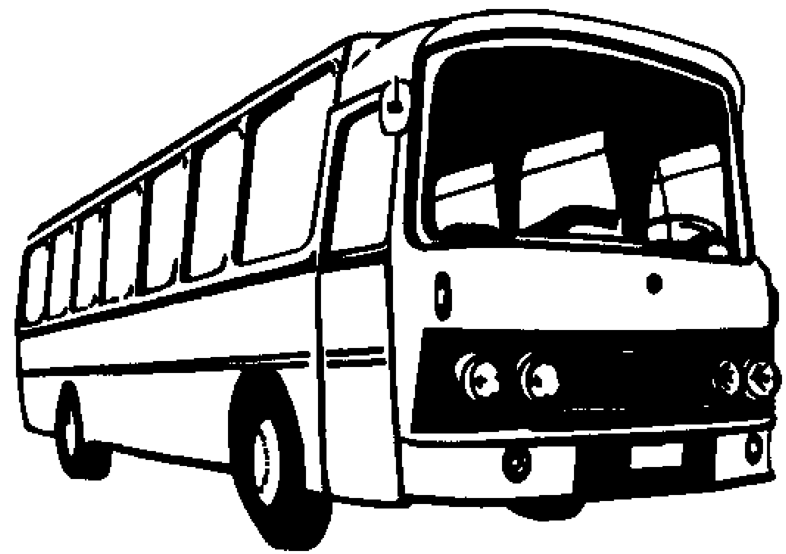




Hilbert’s Grand Hotel

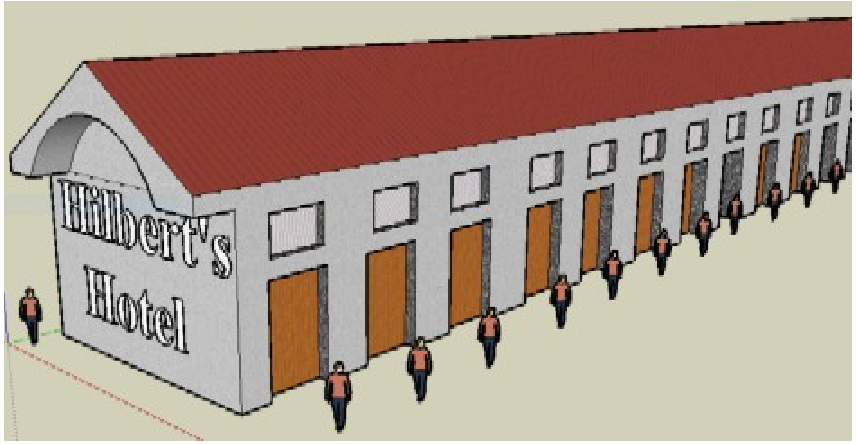
- 259 new arrivals
- Everybody moves up 259 rooms, so if they are in room *n* they move to room *n* + 259
- New arrivals put in rooms 1 to 259. Done!
- Works for any finite k number of new arrivals.
- Correspondence n → n+k
- $k + \aleph_0 = \aleph_0$



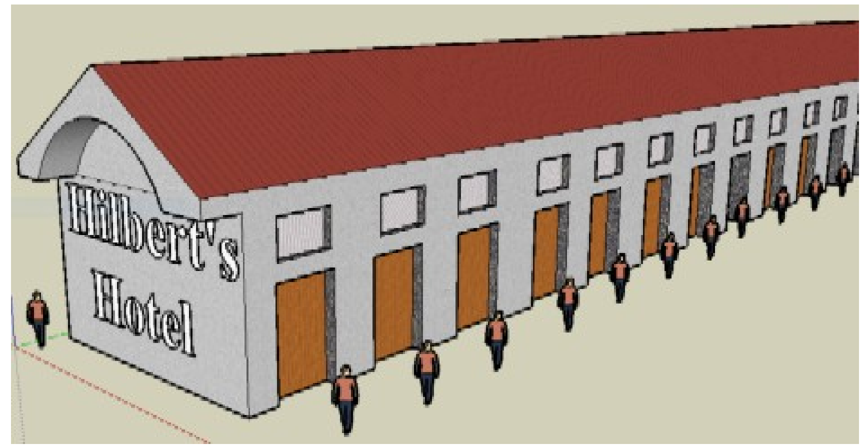


Hilbert's Grand Hotel

Suppose now an infinite long buss arrives fully filled with an infinite number of new arrivals



Hilbert's Grand Hotel



Everybody moves to the room with number twice that of their current room: n ⟶ 2n

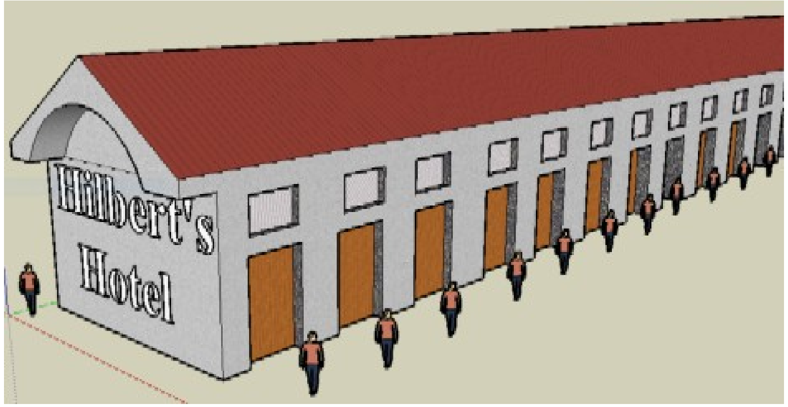
All the odd numbered rooms are now **free** and he uses them to accommodate the infinite number of people on the bus

N:	1	2	3	4	5	...	n	...
	↕	↕	↕	↕	↕		↕	
E:	2	4	6	8	10	...	2n	...

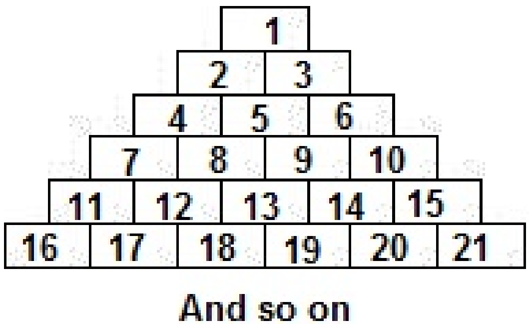
$$\aleph_0 + \aleph_0 = \aleph_0$$

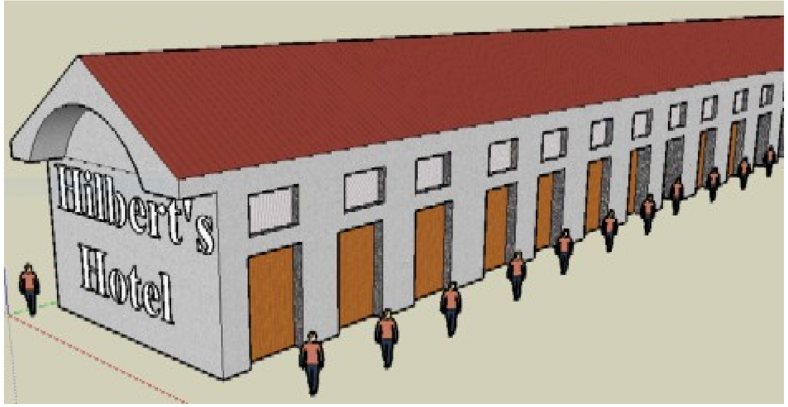
Hilbert's Grand Hotel

Countably infinite number of buses each with countably infinite passengers

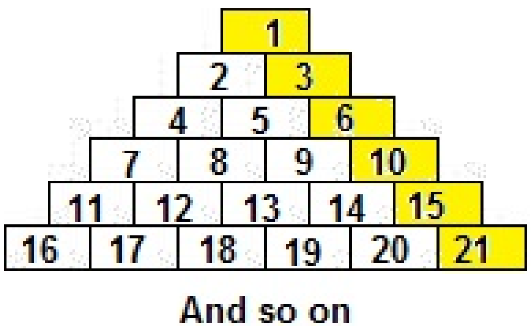


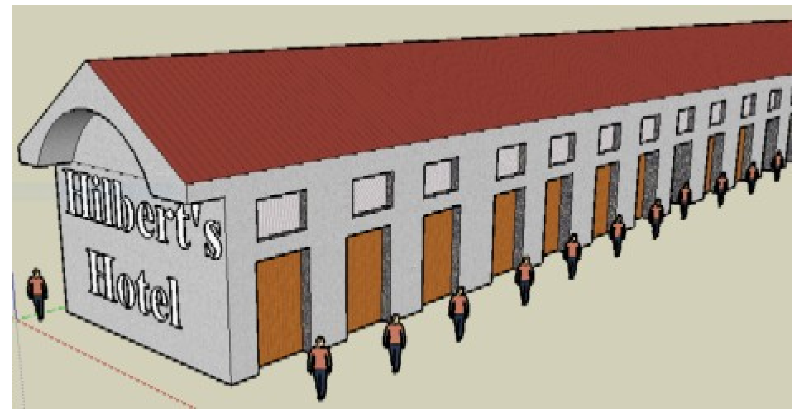
Hilbert’s Grand Hotel



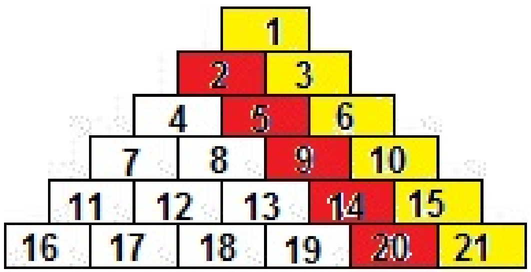


Countably infinite number of buses each with countably infinite passengers

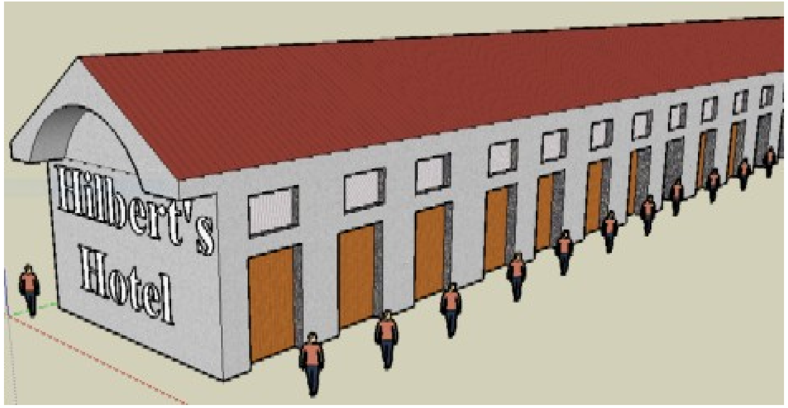




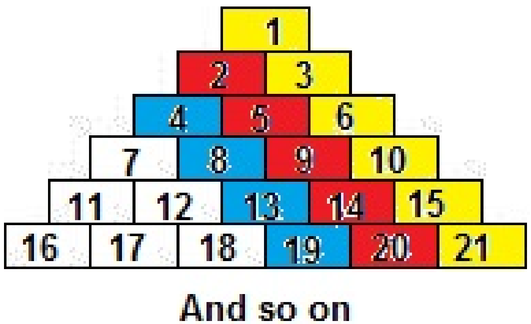
Hilbert’s Grand Hotel

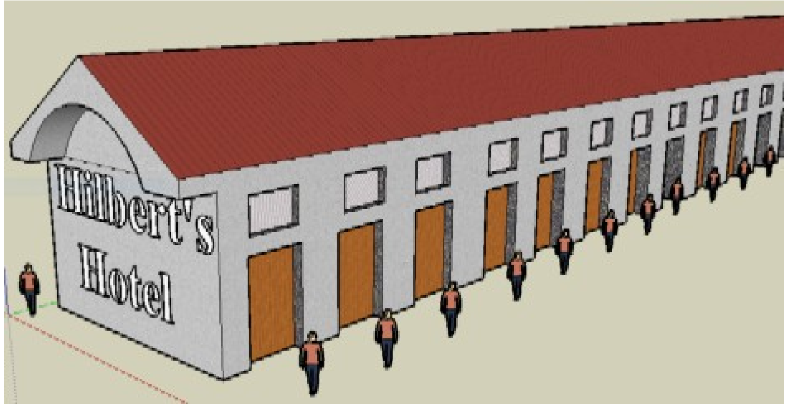


And so on

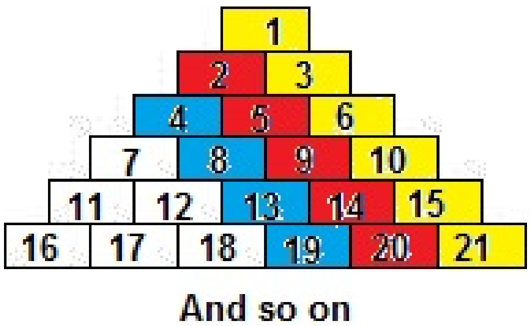


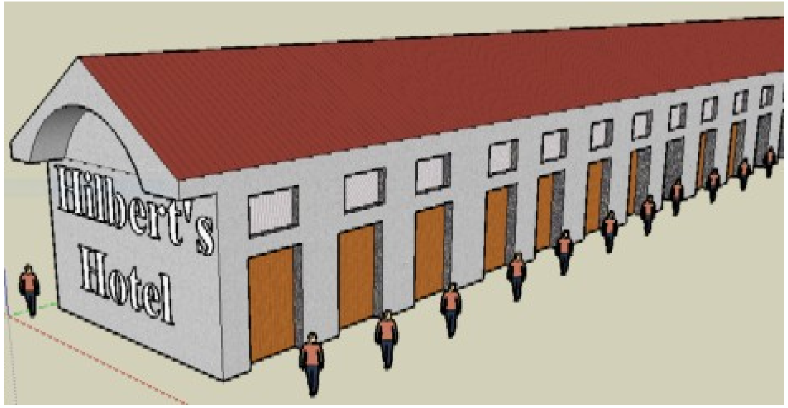
Hilbert’s Grand Hotel





Hilbert's Grand Hotel





$\aleph_0$ times $\aleph_0 = \aleph_0$

Hilbert’s Paradox of the Grand Hotel

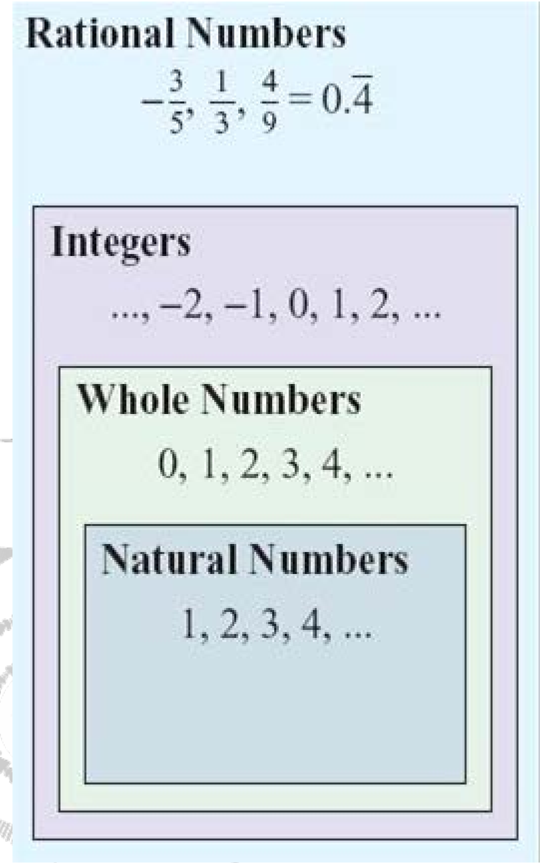
- A more tangible algorithm for selection of room numbers
- Empty the odd numbered rooms by sending the guest in room n to room 2^n
- Put the nth passenger of the first infinite bus load in rooms 3^n
- The nth passenger of the second infinite bus in rooms 5^n
- In general, the nth passenger in bus number k uses the rooms P^n where P is the kth odd prime number.
- This solution leaves certain rooms empty (which may or may not be useful to the hotel); specifically, all odd numbers that are not prime powers, such as 15 or 847, will no longer be occupied.
- But what a *wonderful* discovery, an infinite amount busses filled with infinite passengers, fit into a single infinite hotel!!!
- https://www.youtube.com/watch?v=Uj3_KqkI9Zo

Only one size of infinity?

- I see what might be going on – we can do this because these infinite sets are discrete, have gaps, and this is what allows the method to work because we can somehow interleave them and this is why we always end up with $\aleph_0$.
- Lets fill in the gaps. A rational number or fraction is any integer divided by any nonzero integer, for example, 5/4, 87/32, -567/981.
- The rationals don't have gaps in the sense that between any two rationals there is another rational.
- Even more the rational numbers are dense in the reals, meaning that for every real number there is always a rational number "arbitrarily close" to it.
- Surely the rationals are not countable, right??

Cardinality of the Rational numbers

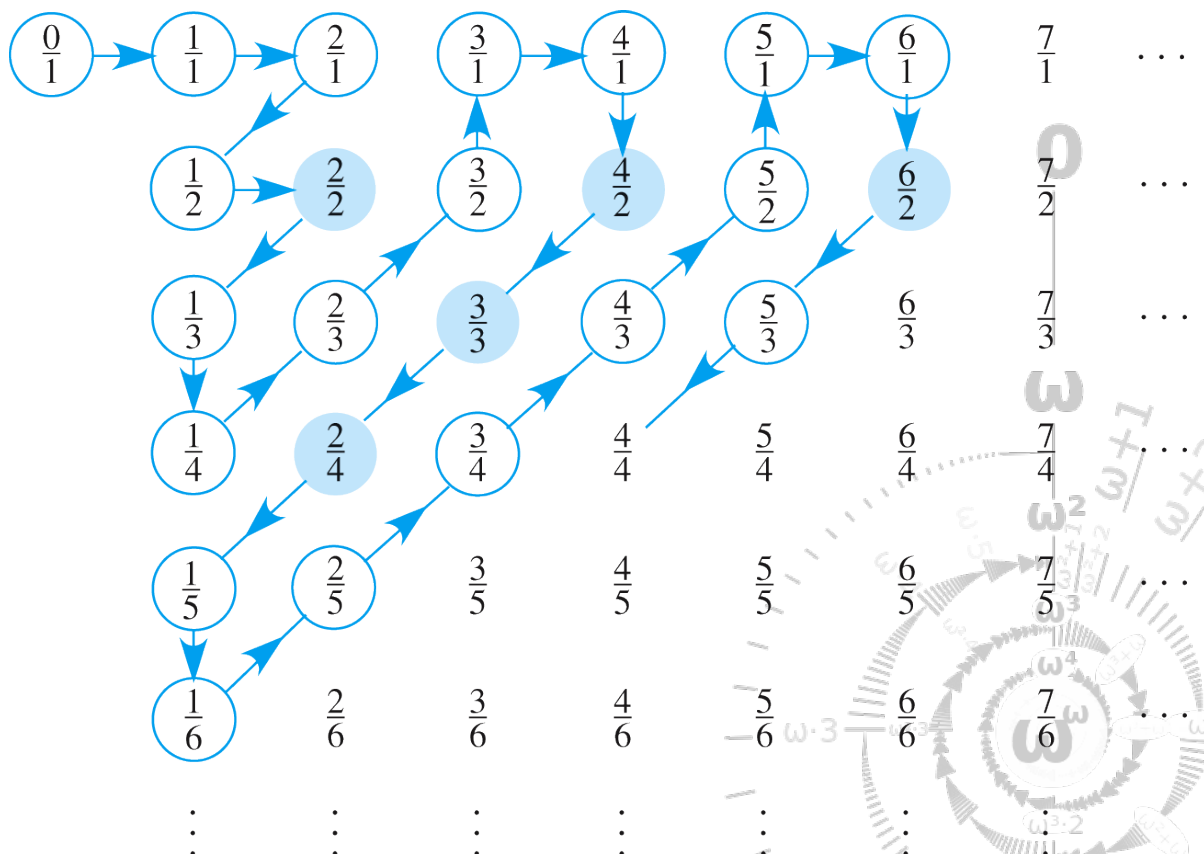
- *Rationals*=all fractions
- $Q = \left\{ \frac{p}{q} \middle| p, q \in Z, q \neq 0 \right\}$
- Is the set of all rationals countable infinite?
- If we can find a one-to-one correspondence with the positive rationals we play the same game as integers to get the negative ones.



Cardinality of the Rational numbers

$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\cdots$
	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\cdots$
	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\cdots$
	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\cdots$
	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\cdots$
	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Cardinality of the Rational numbers



In particular, the one to one correspondence is given by

1↔0/1, 2↔1/1, 3 ↔2/1, 4↔1/2, 5 ↔1/3, 6 ↔1/4,…

So Cardinality of the Rational numbers is $\aleph_0$

Cardinality of the Rational numbers

The function f:Q→N, $f\left(\frac{p}{q}\right) = 2^p 3^q$

Injects the rational into the counting numbers

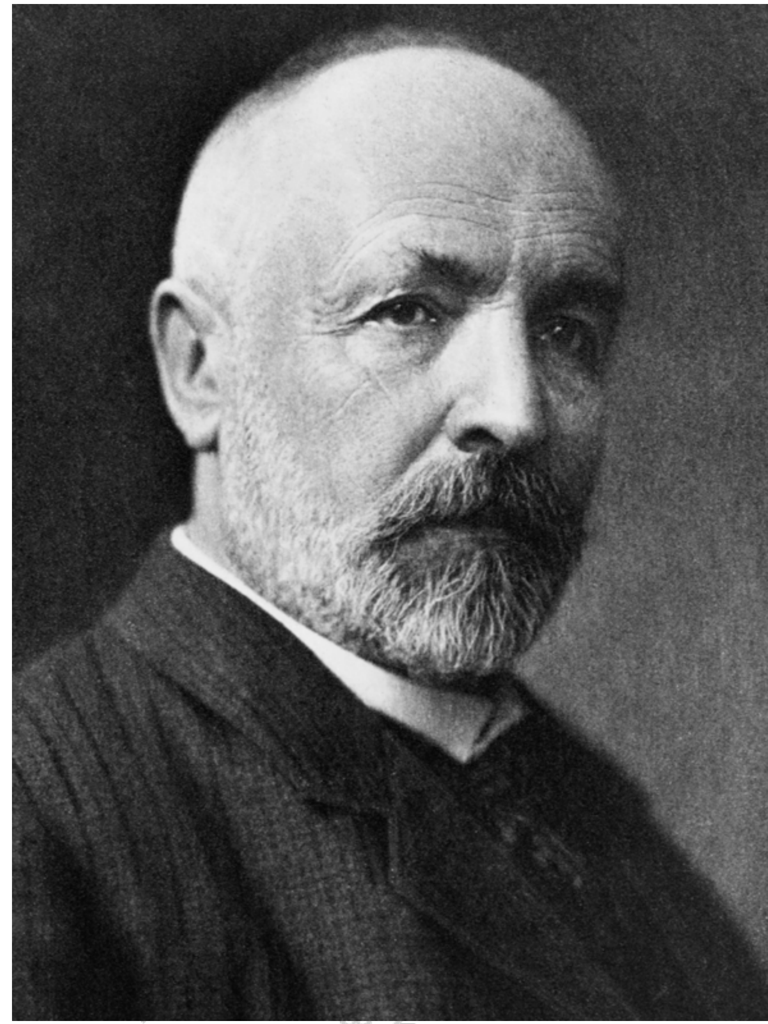
Thm: The following are equivalent

a) Set A is countable infinite
b) There is an onto function f:N →A
c) There is an injection (one-to-one) function f:A →N

Need discrete math or beyond to go deeper into such proofs.

Cantor’s infinities

Bronze monument to Cantor
in Halle-Neustadt



Georg Cantor 1845 – 1918

Uncountability of the set of Real Numbers

So Far all of the sets we have encountered are countable infinity. Is there only one “size” of infinity?

In order to prove that other sized of infinity exist we must show that there is a set of numbers that is not countable. That is bigger than “first size” of infinity.

Suppose that **R** were countable. In particular, any subset of **R**, being smaller, would be countable also. So the interval (0,1) would be countable.

So if we can show that all the decimal numbers between 0 and 1 are not countable we have discovered that they are of a different size of infinity.

Uncountability of R
Cantor's Diabolical Diagonal

Suppose that all real numbers between 0 and 1 are countable. So there exist a pairing with the countable numbers.

That means we can we have a list, were we can see the first number, second, etc.

$$r_1 , r_2 , r_3 , r_4 , r_5 , r_6 , r_7, \ldots$$

Suppose the above list contains ***EVERY*** real number between 0 and 1.

Cantor's diabolical diagonalization argument will take this supposed list, and create a number between 0 and 1 which ***is not*** on the list. This will contradict the countability assumption hence proving that **R** is not countable.

Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.							
r_2	0.							
r_3	0.							
r_4	0.							
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	...
r_2	0.							
r_3	0.							
r_4	0.							
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	...
r_2	0.	1	1	1	1	1	1	...
r_3	0.							
r_4	0.							
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	...
r_2	0.	1	1	1	1	1	1	...
r_3	0.	2	5	4	2	0	9	...
r_4	0.							
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	...
r_2	0.	1	1	1	1	1	1	...
r_3	0.	2	5	4	2	0	9	...
r_4	0.	7	8	9	0	6	2	...
r_5	0.							
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	...
r_2	0.	1	1	1	1	1	1	...
r_3	0.	2	5	4	2	0	9	...
r_4	0.	7	8	9	0	6	2	...
r_5	0.	0	1	1	0	1	0	...
r_6	0.							
r_7	0.							
:								
r_{evil}	0.							

Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	...
r_2	0.	1	1	1	1	1	1	...
r_3	0.	2	5	4	2	0	9	...
r_4	0.	7	8	9	0	6	2	...
r_5	0.	0	1	1	0	1	0	...
r_6	0.	5	5	5	5	5	5	...
r_7	0.							
:								
r_{evil}	0.							

Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	...
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r_4	0.	7	8	9	0	6	2	...
r_5	0.	0	1	1	0	1	0	...
r_6	0.	5	5	5	5	5	5	...
r_7	0.	7	6	7	9	5	4	...
:								
r_{evil}	0.							

Cantor's Diagonalization Argument

← Decimal expansions of r_i →

r_1	0.	1	2	3	4	5	6	...
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r_3	0.	2	5	4	2	0	9	...
r_4	0.	7	8	9	0	6	2	...
r_5	0.	0	1	1	0	1	0	...
r_6	0.	5	5	5	5	5	5	...
r_7	0.	7	6	7	9	5	4	...
:								
r_{evil}	0.	2	9	7	5	0	4	...

Is New decimal number in our original list?

r_{evil}	0.	2	9	7	5	0	4	...

r_{evil} Is constructed by changing the digits of the main diagonal number.

Is New decimal number in our original list?

r_{evil}	0.	2	9	7	5	0	4	...

Is this the same as the first number in our list?
NO, we changed the first digit.
It cannot be the second number since we changed the second digit.
It cannot be the nth number in the list since we changed the nth digit.

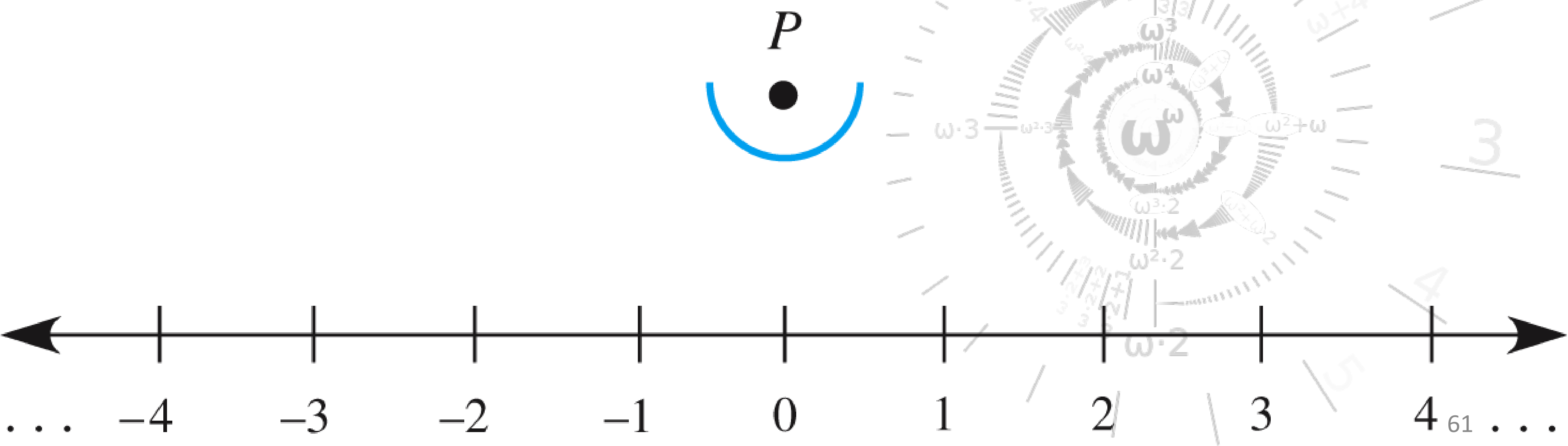
Conclusion: This new decimal number is not in our list. But it is still a decimal number between 0 and 1.

So all the real numbers between 0 and 1 are UNCOUNTABLE

Eureka!, you have found a **new** "size" of infinity.

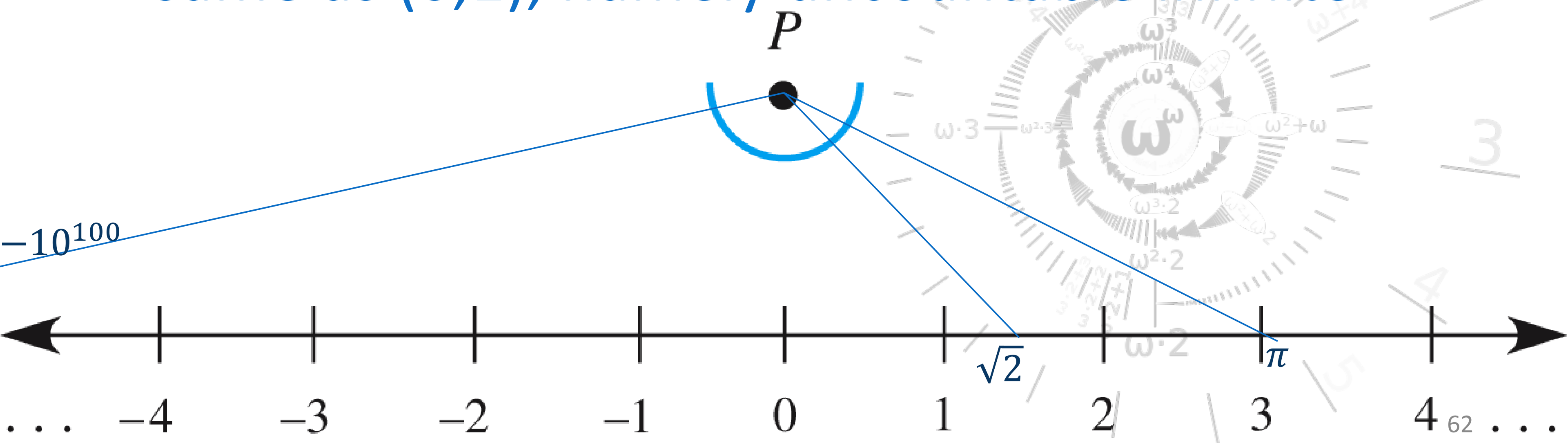
Cardinality of the real line

- If the cardinality of all real numbers between (0,1) is uncountable, what about all real numbers?
- Let the decimals in the interval (0,1) be represented as a line segment and bend into a semicircle, position it above the real line:



Cardinality of the real line

- Rays emanating from point P will establish a geometric pairing for the points on the semicircle with the points on the line.
- So the cardinality of the real number line is same as (0,1), namely uncountable infinite

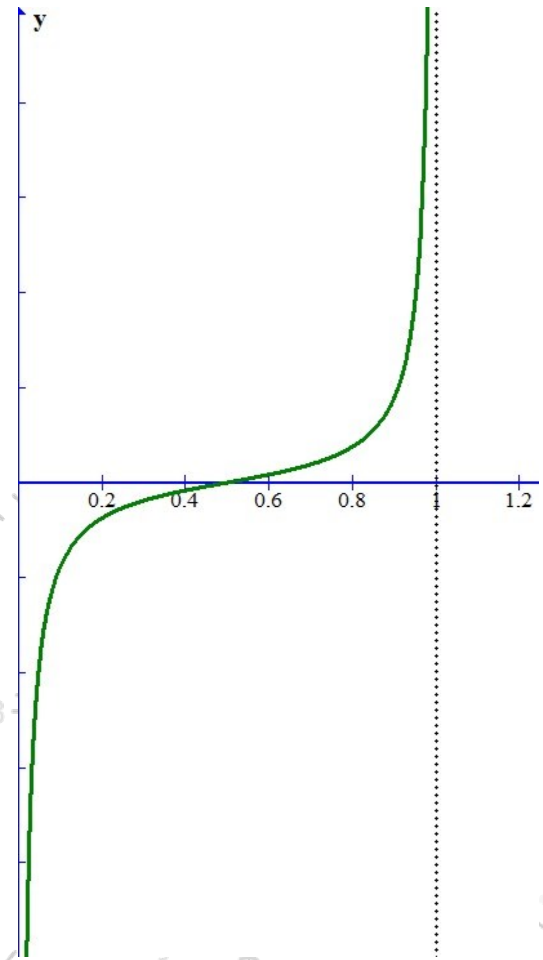


The cardinality of the reals is the same as that of the interval of the reals between 0 and 1

The cardinality of the reals is often denoted by *c*
for the continuum of real numbers.

Consider the one-to-one correspondence: f:(0,1)→ ℝ

$$f(x) = \frac{2x - 1}{x - x^2}$$



Power set of a set

Given a set A, the *power set* of A, denoted by P(A), is the set of all subsets of A.

For example, for set A = {a, b, c}, has cardinality n(A)=3, and has 2^3 =8 subsets, namely:

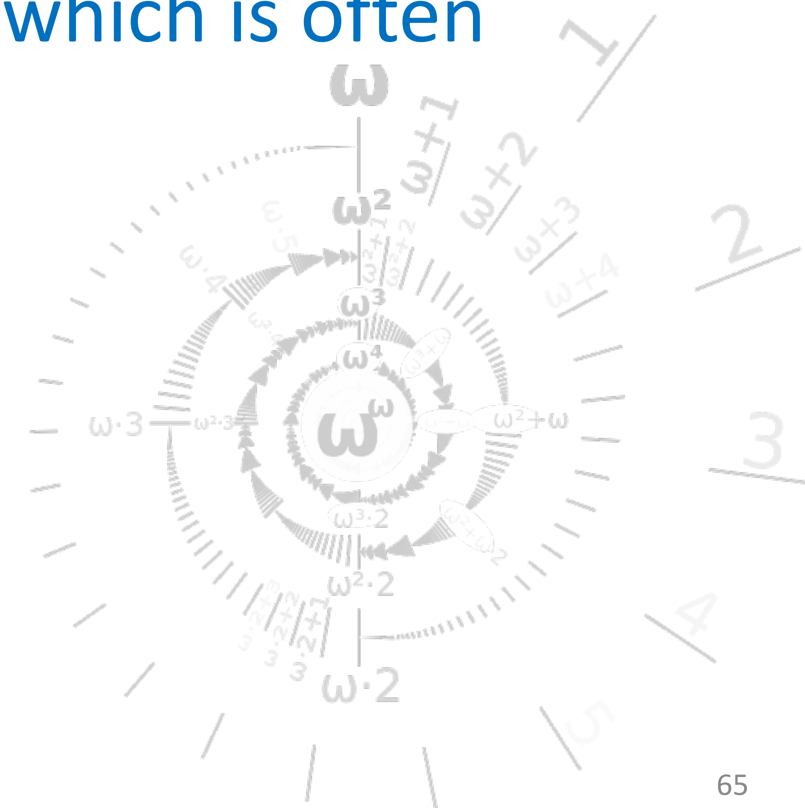
P(A) = { { }, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }

{ } is the empty set and if a set has *n* elements it has 2^n subsets. So the cardinality of the power set is: $n(P(A)) = 2^{n(A)}$

The power set is itself a set

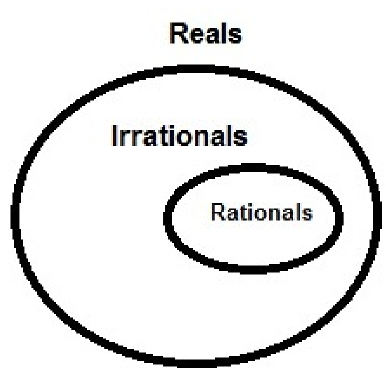
$$c = 2^{\aleph_0}$$

Indeed we can show that the reals (the continuum) have the cardinality of the power set of the natural numbers which is often written as above.



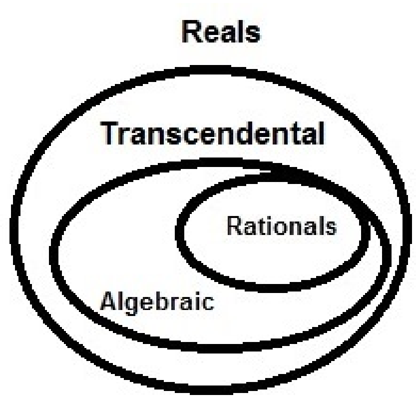
Cardinality of some sets

Set	Description	Cardinality
Natural numbers	1, 2, 3, 4, 5, …	$\aleph_0$
Integers	…, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, …	$\aleph_0$
Rational numbers or fractions	All the decimals which terminate or repeat	$\aleph_0$
Irrational numbers	All the decimals which do not terminate or repeat	C
Real numbers	All decimals	C



Cardinality of some sets

Set	Description	Cardinality
Real numbers	All decimals	c
Algebraic numbers	All solutions of polynomial equations with integer coefficients. All rationals are algebraic as well as many irrationals e.g. $\sqrt{2}$.	$\aleph_0$
Transcendental numbers	All reals which are not algebraic numbers e.g. π, e, $2^{\sqrt{2}}$	c



$\boldsymbol{n}(\mathbb{R}^n) = \boldsymbol{n}(\mathbb{R})$

- **"I see it, but I don't believe it"**. Cantor
- If we can show a one-to-one correspondence between $(0,1) \times (0,1) \times \cdots (0,1) \leftrightarrow (0,1)$, we can extend this result for $\mathbb{R}^n \leftrightarrow \mathbb{R}$, proving that $\boldsymbol{n}(\mathbb{R}^n)$=c=$\boldsymbol{n}(\mathbb{R})$
- Let $x \in (0,1)^n$ decimal expansion be given by: $(0.a_1a_2a_3 \ldots, 0.b_1b_2b_3 \ldots, \cdots, 0.z_1z_2z_3 \ldots)$, then we can match it to y $\in (0,1)$ by interleaving digits ("chunks") as $(0.a_1b_1c_1 \ldots z_1a_2b_2c_2 \ldots z_2a_3b_3c_3 \ldots z_3 \ldots)$

$\boldsymbol{n}(\mathbb{R}^{\boldsymbol{n}}) = \boldsymbol{n}(\mathbb{R})$

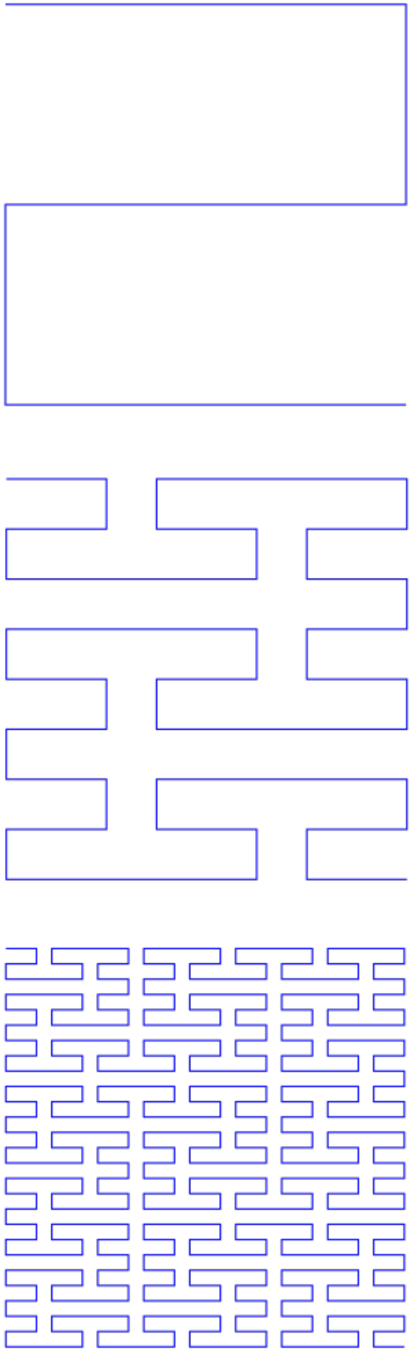
- Cantor originally tried interleaving the digits himself, but Dedekind pointed out the problem of nonunique decimal representations, namely how ½=0.5=0.4999…., or 0.999….=1
- In order to resolve the interleaving digits not being reversible in those cases, we need to use "chunking" by always choosing the non terminating zero representation of the decimal, meaning 0.199… instead of its equal representation 0.200…, and going to the next nonzero digit, inclusive.
- For example For example, 1/200=0.00499… is broken up as 004 9 9 9…, and 0.01003430901111… is broken up as

  01 003 4 3 09 01 1 1…
- Now instead of interleaving digits, we interleave chunks. To interleave 0.004999… and 0.01003430901111…, we get

  0.004 01 9 003 9 4 9….

$$\boldsymbol{n}(\mathbb{R}^{\boldsymbol{n}}) = \boldsymbol{n}(\mathbb{R})$$

- Lets visualize this extraordinary result
- The result was proved by Cantor in 1878, but only became intuitively apparent in 1890, when Giuseppe Peano introduced the space-filling curves.
- Curved lines that twist and turn enough to fill the whole of any square, or cube, or hypercube, or finite-dimensional space.
- Pictured on the right are the first three steps of a fractal construction whose limit is a space-filling curve, showing that there are as many points in a one-dimensional line as in a two-dimensional square.



Cantor’s Theorem

- For any set A, the power set of A, P(A) has a **strictly greater cardinality** than A itself.
- Clearly we can inject A→P(A), by taking each element x →{x} to the singleton set containing it.
- So n(A)≤n(P(A)), but now we need to show equality cannot happen and thus strict inequality exist.
- So in order to show no one-to-one correspondence exist, it suffices to show no onto function exist.

Cantor’s Theorem

- **Complete proof of Cantor’s power set theorem**
  - $Consider\ f: A \to P(A). Then\ \{x \in A | x \notin f(x)\} \notin f(A). Q.E.D.$
- Feel free to use it to impress family and friends.
- Not clear?
- Ok, lets break this proof down for us mere mortals.

No set can be placed in one-to-one correspondence with its power set

Proof by contradiction. Suppose there is such onto function from A→P(A), and correspondence given by:

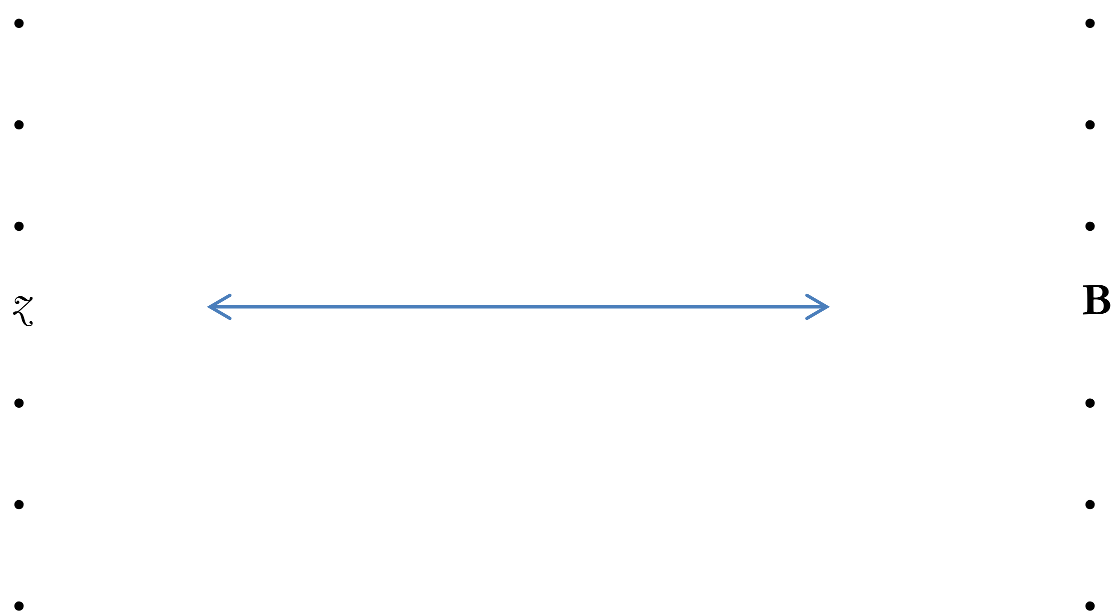
Elements of A		Elements of P[A] (i.e. subsets of A)
a	↔	$\{c, d\}$
b	↔	$\{e\}$
c	↔	$\{b, c, d, e\}$
d	↔	$\{\ \}$
e	↔	A
f	↔	$\{a, c, e, g, \ldots\}$
g	↔	$\{b, k, m, \ldots\}$
.		.
.		.
.		.

No set can be placed in one-to-one correspondence with its power set

Elements of A		Elements of P[A] (i.e. subsets of A)
a	⟷	{*c*, *d*}
b	⟷	{*e*}
c	⟷	{*b*, *c*, *d*, *e*}
d	⟷	{ }
e	⟷	A
f	⟷	{*a*, *c*, *e*, *g*, …}
g	⟷	{*b*, *k*, *m*, …}
.		.
.		.
.		.

Let **B** be the set of each and every element of the original set A that is not a member of the subset with which it is matched.
For the matching above, **B** = *{a, b, d, f, g, …}*

Now B is just a subset of A so must appear somewhere in the right-hand column and so is matched with some element of A say *z*



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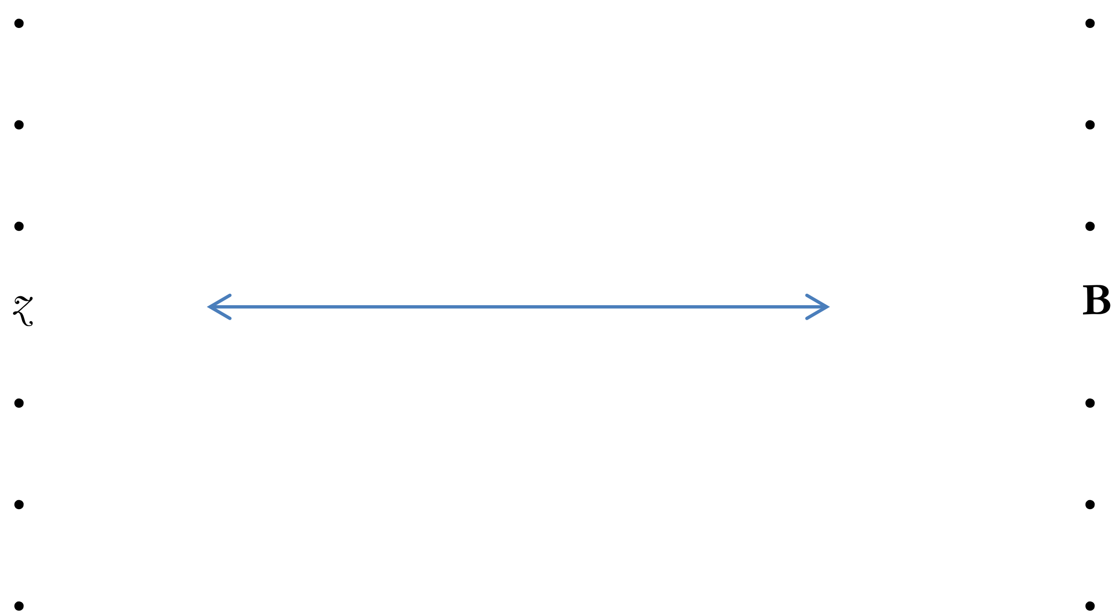
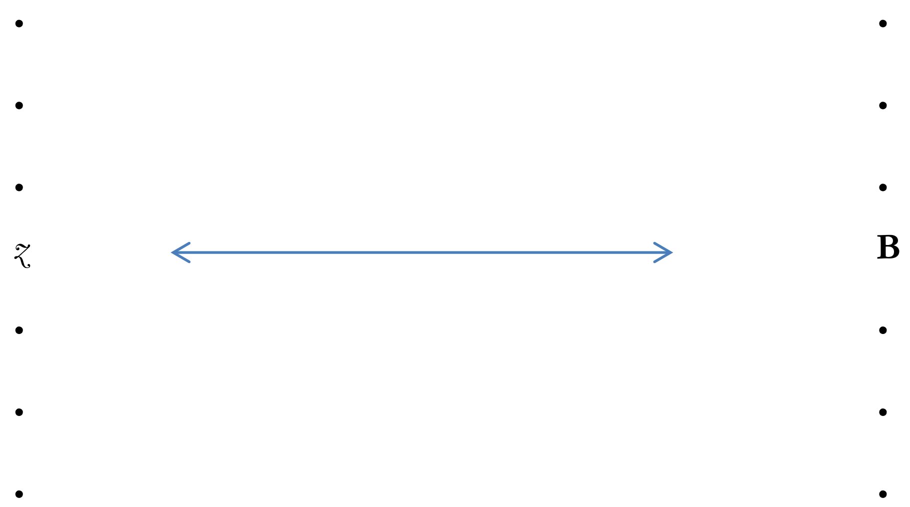


Now for the fatal question! Is *z* an element of **B**?

Since *z* is an element of A, it must be in B or not be in B

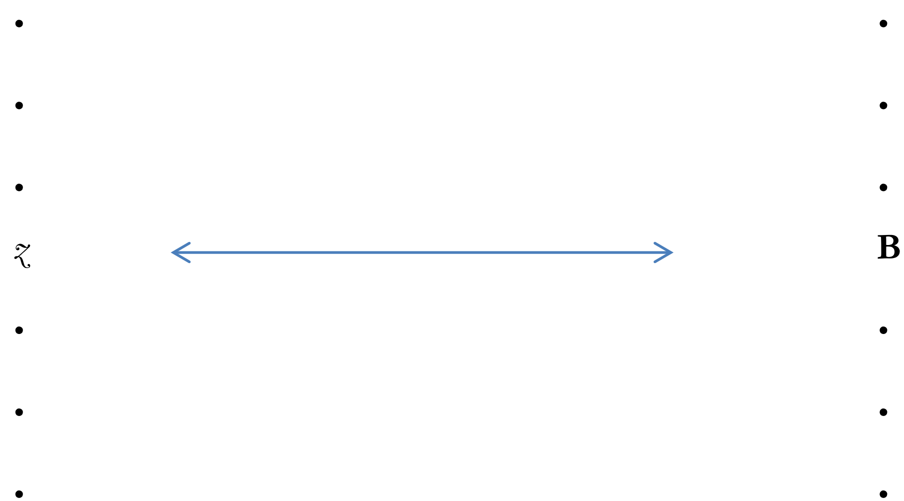
Case 1: Suppose *z* *is* an element of B

Then by definition of set B, which consists of elements which do not belong to their matching subset, z **must not** belong to B! Contradiction



Case 2: Suppose *z is not* an element of B

Then *z* satisfies the defining property of B which is that it consists of elements which do not belong to their matching subset, so *z* **does belong** to B! Contradiction!



Thus proving, that for any set A, the power set of A, P(A) has a **strictly greater cardinality** than A itself.

Infinity of infinities

Reals have smaller cardinality than the power set of the reals.

Which is smaller than the power set of the power set of the reals

Which is smaller than the power set of the power set of the power set of the reals

Which is smaller than the power set of the power set of the power set of the power set of the reals, etc!

$$n(\mathbb{R}) < n(P(\mathbb{R})) < n\left(P(P(\mathbb{R}))\right) < n\left(P\left(P(P(\mathbb{R}))\right)\right) < \cdots$$

“Infinitely many more cardinals”

The continuum Hypothesis

- So far, $n(\mathbb{N}) = \aleph_0$ and $n(\mathbb{R}) = c$, called the continuum.
- The proposal originally made by Georg Cantor that there is no infinite set with a cardinal number between that of the "small" infinite set of integers $\aleph_0$ and the "large" infinite set of real numbers c (the "continuum"). Symbolically, the continuum hypothesis is that $\aleph_1 = c$

Continuum hypothesis

The Continuum hypothesis states:

there is no transfinite cardinal falling ***strictly*** between $\aleph_0$ and *c*

Work of Gödel (1940) and of Cohen (1963) together implied that the continuum hypothesis was independent of the other axioms of set theory

Cantor's assessment of his theory of the infinite

My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all because I have followed its roots, so to speak, to the first infallible cause of all created things.

Cantor circa 1870

Transfinite Arithmetic

- When Cantor proposed his cardinal numbers (informally, the sizes of all finite or infinite sets), he also devised a new type of arithmetic for these generalized numbers.

- If the cardinal numbers are **finite**, then the operations are those of **ordinary arithmetic**.

- If the cardinal numbers are transfinite, then we have the following identities:

$$\aleph_a + \aleph_b = \max\{\aleph_a, \aleph_b\}$$

$$\aleph_a \cdot \aleph_b = \max\{\aleph_a, \aleph_b\}$$

$$2^{\aleph_n} = \aleph_{n+1} > \aleph_n$$ (Cantor's **Power Set Theorem**)

Hence we have, for example

$$\aleph_0 + \aleph_0 = \aleph_0 , \quad \aleph_0 + c = c$$

$$\aleph_0 \cdot \aleph_0 = \aleph_0 , \quad \aleph_0 \cdot c = c$$

Historical Background (Pre-Cantor)

- ***5th Century B.C.*** The Greek philosopher **Zeno** of Elea proposes his paradoxes of motion, all rooted in some misconceptions of the infinite (but also in deeper questions relating to the nature of time and space...)

- ***4th Century B.C.*** **Aristotle's** work in metaphysics makes a distinction between the "potential" infinite and the "actual" infinite.

- ***Early 17th Century*** **Galileo** shows the equinumerosity of the natural numbers with the perfect squares, leading to his celebrated paradox. His attempt to resolve this problem launches the first modern line of inquiry towards the infinite.

- ***Late 17th Century*** **Isaac Newton** and **Gottfried Leibniz** independently develop the infinitesimal calculus, effectively paving the way for abstract analysis – a pillar of modern mathematics.

Historical Background (Post-Cantor)

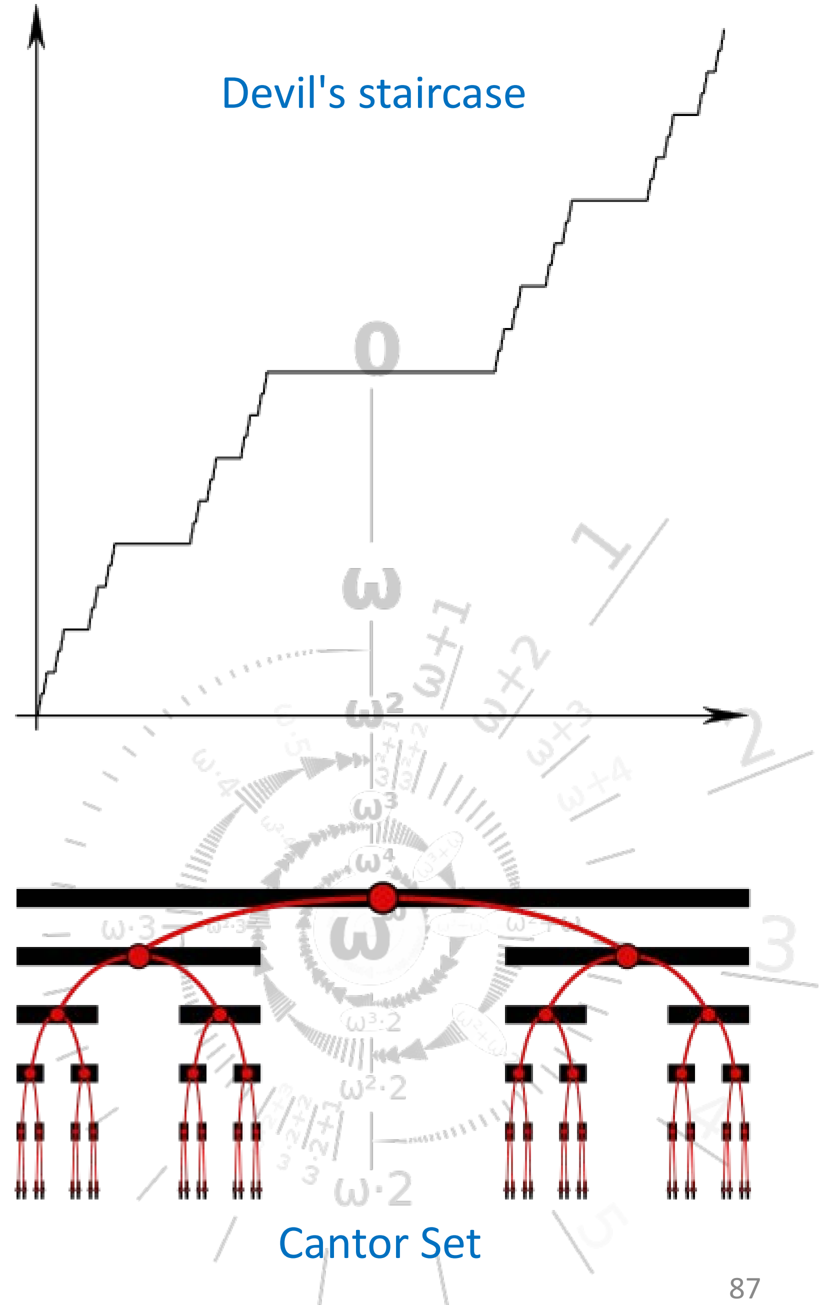
- ***1870's*** The Russian mathematician **Georg Cantor** proposes his ground-breaking theory of sets and an arithmetic for "transfinite" cardinal numbers. His work grounds the concept of infinity on a rigorous mathematical basis and equips mathematics with a firm logical foundation.

- ***1900's*** The British logician **Bertrand Russell** points to the simplest and most damaging paradox that emerges from Cantorian set theory.

- ***1920's*** The German mathematicians **Ernst Zermelo** and **Abraham Fraenkel** formulate an axiomatic theory of sets (known as ZFC when including the axiom of choice) and resolve all standing paradoxes.

- ***1960's* -** The American mathematician **Paul Cohen** builds on previous work by the Austrian logician **Kurt Gödel** to show that the continuum hypothesis is independent from ZFC.

"No one shall expel us from the Paradise that Cantor has created."
David Hilbert
1926



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